5.3 Continuous Functions on Intervals

Def'n. A function $f: A \to \mathbb{R}$ is said to be bounded on A if there is a constant $M > 0$ such that

$|f(x)| \le M$, for all $x \in A$.

Ex $f(x) = \frac{1}{x}$ is not bounded on $(0, 1]$

The Boundedness Thm.

Thm. Let $I = [a,b]$ and let

$f: I \to \mathbb{R}$ be continuous on I.

Then f is bounded on I.

Pf. (by Contradiction)

Suppose f is not bounded.

Then for every integer

$n \in \mathbb{N}$, there is a point x_n in I

such that $|f(x_n)| > n$.

Since I is bounded,

the sequence $X = (x_n)$ is bounded. Hence the Bolzano-Weierstrass Theorem implies there is a subsequence $X' = (x_{n_k})$ of X that converges to a number x. Since I is closed and the elements of X' belong to I, it follows that $x \in I$.

In fact, Theorem 3.2.6

states that if $\lim(y_n) = y$

and $a \le y \le b$, then

$$a \le \lim(y_n) \le b.$$

Thus, f is continuous at x, $(f(x_{n_r}))$ converges to $f(x)$. We then conclude that the convergent sequence $(f(x_{n_r}))$ must be bounded. But this is a contradiction, since

$$|f(x_{n_r})| > n_r \geq r, \quad \text{all } r \in \mathbb{N}$$

This proves the Boundedness Theorem.

Def'n. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. We say that f has an absolute maximum on A if there is a point $x' \in A$ so that

$$f(x') \geq f(x), \qquad \text{for all } x \in A.$$

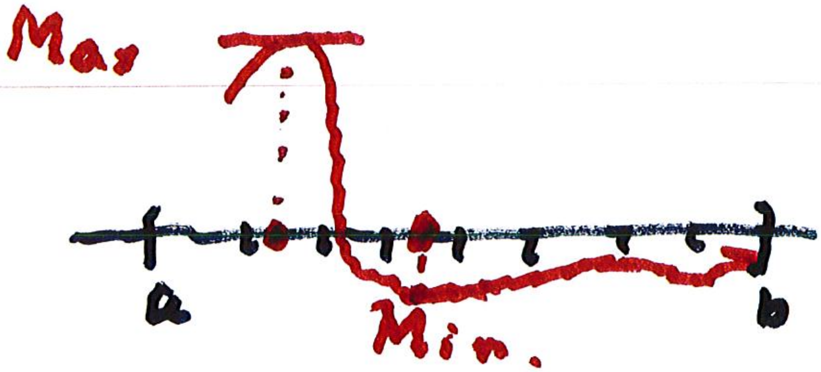
Similarly, f has an absolute minimum on A if there is a point $x'' \in A$ such that

$$f(x'') \leq f(x), \qquad \text{for all } x \in A.$$

Maximum-Minimum Theorem.

Let $I = [a, b]$ and let $f: I \to \mathbb{R}$ be continuous on I.

Then f has an absolute maximum and an absolute minimum on I



Proof: Consider the set $f(I) = \{ f(x) : x \in A \}$.

Let $s' = \sup f(I)$ and let $s'' = \inf f(I)$. We will show that there exist points x' and x'' such that $s' = f(x')$ and $s'' = f(x'')$

We do this for x'. (The case for x'' is similar).

Since $S' = \sup f(I)$,

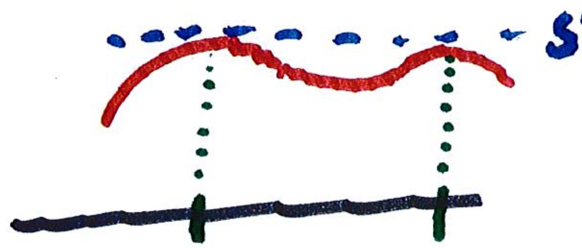
(The Boundedness Thm $\rightarrow S'$ is finite.)

We want to show that

there exists at least one

number x' so that

$f(x') = S'$.



If not,

there exists no number x

satisfying $f(x) = S'$.

Consider the function

$$g(x) = \frac{1}{s' - f(x)}.$$

We already know $f(x) \leq s'$ for all x and and that there is no number x with $f(x) = s'$

Hence $s' > f(x)$, for all $x \in I$.

Hence the function

$$g(x) = \frac{1}{s' - f(x)} > 0 \quad \text{for all } x \in I.$$

By the Quotient Rule, the function g is continuous at each point $x \in I$. By the above result there is a positive number M so that g is bounded, i.e.,

$$\frac{1}{s' - f(x)} < M$$

$$\rightarrow \quad \frac{1}{M} < s' - f(x)$$

$$\rightarrow \quad f(x) < s' - \frac{1}{M}.$$

which shows that $s' - \frac{1}{M}$ is an upper bound, which shows that $\sup f(I) \le s' - \frac{1}{M}$,

which contradicts the

statement that $\sup f(I) = s'$

It follows that there must be

at least one value x' such that

$f(x') = s'$.

We conclude that if f is continuous

on $[a,b]$, then f has at least one

absolute maximum.

Our third theorem is Bolzano's "Intermediate Value Thm.

If $I = [a, b]$, let

$f: I \to \mathbb{R}$ be continuous

on I. If $f(a) < 0 < f(b)$

$\{$ or $f(a) > 0 > f(b) \}$

then there exists a number

$c \in (a,b)$ so that $f(c) = 0$.

Proof. We assume that

$f(a) < 0 < f(b)$. Let

$I_1 = [a_1, b_1]$, where

$a_1 = a$, and $b_1 = b$. We let

$p_1 = \dfrac{a_1 + b_1}{2}$. If $f(p_1) = 0$,

we take $c = p_1$ and we are

done. If $f(p_1) \neq 0$, then

either $f(p_1) > 0$ or $f(p_1) < 0$.

In the first case, set $a_2 = a_1$ and set $b_2 = p_1$. In the second case (when $f(p_1) < 0$), set $a_2 = p_1$ and set $b_2 = b_1$. In both cases, we have $f(a_2) < 0$ and $f(b_2) > 0$.

We continue this bisection process. Assume that intervals $I_1 \supset I_2 \supset \cdots I_k$ have been obtained by succesive bisection and that

$f(a_k) < 0 < f(b_k)$.

We set $p_k = \frac{1}{2}(a_k + b_k)$.

If $p_k = 0$, we take $c = p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} = a_k$ and $b_{k+1} = p_k$.

If $f(p_k) < 0$, we set $a_{k+1} = p_k$ and $b_{k+1} = b_k$.

In either case, we let

$I_{k+1} = [a_{k+1}, b_{k+1}]$

Then $I_{k+1} \subset I_k$ and

$f(a_{k+1}) < 0$ and $f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done.

If the process does not terminate, then we have a nested sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that

$f(a_n) < 0 < f(b_n)$.

The intervals are obtained by repeated bisection, so that the length of I_n equals $b_n - a_n = \frac{b-a}{2^{n-1}}$.

Let c be any point belonging to to I_n for all n. It satisfies

$a_n \le c \le b_n$, so

we have

$$0 \le c - a_n \le b_n - a_n = \frac{(b-a)}{2^{n-1}},$$

and

$$0 \le b_n - c \le b_n - a_n = \frac{(b-a)}{2^{n-1}}.$$

The Squeeze Theorem implies

that $\lim(a_n) = c = \lim(b_n)$

Since f is continuous at c,

we have

$\lim(f(a_n)) = f(c) = \lim(f(b_n))$.

The fact that $f(a_n) < 0$

for all $n \in \mathbb{N}$ implies that

$f(c) = \lim(f(a_n)) \leq 0$.

Also, the fact $f(b_n) > 0$

implies that

$f(c) = \lim(f(b_n)) \geq 0$.

We conclude that $f(c) = 0$.

This proves the Intermediate Theorem

when $f(a) < f(b)$.

Bolzano's Intermediate Value Thm:

Suppose that I is an interval

and let $f: I \longrightarrow \mathbb{R}$

be continuous on I. If

$a, b \in I$ and if $k \in \mathbb{R}$

satisfies $f(a) < k < f(b)$, then

there exists a point c with

$a < c < b$ such that $f(c) = k$.