Our third theorem is Bolzano's "

Intermediate Value Thm.

If $I = [a, b]$, let

$f: I \to \mathbb{R}$ be continuous

on I. If $f(a) < 0 < f(b)$

($\text{or } f(a) > 0 > f(b)$)

then there exists a number

$c \in (a,b)$ so that $f(c) = 0$.

Proof. We assume that $f(a) < 0 < f(b)$. Let $I_1 = [a_1, b_1]$, where $a_1 = a$, and $b_1 = b$. We let

$$p_1 = \frac{a_1 + b_1}{2}.$$

If $f(p_1) = 0$, we take $c = p_1$ and we are done. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$.

In the first case, set $a_2 = a_1$ and set $b_2 = p_1$. In the second case (when $f(p_1) < 0$), set $a_2 = p_1$ and set $b_2 = b_1$. In both cases, we have $f(a_2) < 0$ and $f(b_2) > 0$.

We continue this bisection process. Assume that intervals $I_1 \supset I_2 \supset \cdots I_k$ have been obtained by succesive bisection and that

$$f(a_k) < 0 < f(b_k).$$

We set $p_k = \frac{1}{2}(a_k + b_k)$.

If $p_k = 0$, we take $c = p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} = a_k$ and $b_{k+1} = p_k$.

If $f(p_k) < 0$, we set $a_{k+1} = p_k$ and $b_{k+1} = b_k$.

In either case, we let $I_{k+1} = [a_{k+1}, b_{k+1}]$

Then $I_{k+1} \subset I_k$ and

$f(a_{k+1}) < 0$ and $f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done.

If the process does not terminate, then we have a nested sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that

$f(a_n) < 0 < f(b_n)$.

The intervals are obtained by repeated bisection, so that the length of I_n equals $b_n - a_n = \frac{b-a}{2^{n-1}}$.

Let c be any point belonging to to I_n for all n. It satisfies

$$a_n \le c \le b_n,$$ so

we have

$$0 \le c - a_n \le b_n - a_n = \frac{(b-a)}{2^{n-1}}.$$

and

$$0 \le b_n - c \le b_n - a_n = \frac{(b-a)}{2^{n-1}}.$$

The Squeeze Theorem implies

that $\lim(a_n) = c = \lim(b_n)$

Since f is continuous at c,

we have

$\lim(f(a_n)) = f(c) = \lim(f(b_n))$.

The fact that $f(a_n) < 0$

for all $n \in \mathbb{N}$ implies that

$f(c) = \lim(f(a_n)) \leq 0$.

Also, the fact $f(b_n) > 0$

implies that

$f(c) = \lim(f(b_n)) \geq 0$.

We conclude that $f(c) = 0$.

This proves the Intermediate Theorem

when $f(a) < f(b)$.

Bolzano's Intermediate Value Thm:

Suppose that I is an interval and let $f: I \longrightarrow \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point c with $a < c < b$ such that $f(c) = k$.

Application of the Intermediate

1. Thm. #8, pg. 140

Let $f(x) = 2\ln x + \sqrt{x} - 2$

Note that

$f(1) = 0 + \sqrt{1} - 2 < 0$

$f(2) = 2\ln 2 + \sqrt{2} - 2$

$> 2 \cdot \frac{1}{2} + 1 - 2 > 0.$

$\therefore$ There is $c \in (1, 2)$ with

$f(c) = 0.$

2. # 17, pg. 141.

Suppose $f: [0,1] \to \mathbb{R}$ is continuous and has only rational values. Then f is constant.

Pf. Suppose f is not constant. Then there are numbers $x_1, x_2 \in [0,1]$ such that $f(x_1) \neq f(x_2)$

By Bolzano's Thm.,

for any irrational number k between $f(x_1)$ and $f(x_2)$, there is a number $c \in [0,1]$ such that $f(c) = k$. (We know there is an irrational number k between $f(x_1)$ and $f(x_2)$ because the irrationals are dense.) Thus the values of f ~~was~~ includes an irrational, which contradicts the hypothesis. Hence, by contradiction, f is constant.

Thm. 3.2.10, pg. 68. The function $\sqrt{x}$ is continuous on $[0, \infty)$.

To prove this theorem, we

make use of 5.1.3, the Sequential Criterion, which states that for every sequence (x_n) in A such that (x_n) converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Proof of Thm 3.2.10. Let

$X = (x_n)$ be a sequence of

real numbers that converges

to x, and suppose that

$x_n \geq 0$. We now consider

the two cases (i) $x = 0$ and

(ii) $x > 0$.

Case (i). If $x = 0$, let $\varepsilon > 0$.

Since $x_n \to 0$, there exists a number $K \in \mathbb{N}$ such that if $n \geq K$, then

$$0 \leq x_n < \varepsilon^2 .$$

Therefore $0 \leq \sqrt{x_n} < \varepsilon$,

for all $n \geq K$.

Since $\varepsilon > 0$ is arbitrary, this implies that $\sqrt{x_n} \to 0$.

Case (ii) If $x > 0$, then $\sqrt{x} > 0$

and we note that

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}$$

$$= \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$, it

follows that

$$|\sqrt{x_n} - \sqrt{x}| \leq \left(\frac{1}{\sqrt{x}}\right)|x_n - x|.$$

The convergence of

$\sqrt{x_n} \to \sqrt{x}$ follows from the

fact that $x_n \to x$.

Since this true for <u>every</u>

sequence, the above Criterion

implies that $\sqrt{x}$ is continuous

at all $x \in [0, \infty)$.