p. 129 #6

1. Let $A \subset \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \varepsilon$.

Pf. For a given $\varepsilon > 0$, choose

$\delta_1 > 0$ so that

if $x \in A$ and $|x - c| < \delta_1$, then

$|f(x) - f(c)| < \frac{\varepsilon}{2}$.

The same applies to y, so

$|f(y) - f(c)| < \frac{\varepsilon}{2}$.

By the Triangle Inequality,

so we get

$$|f(x) - f(y)| = |(f(x) - f(c)) + (f(c) - f(y))|$$

$$\leq |f(x) - f(c)| + |f(y) - f(c)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \checkmark$$

2. p. 129 #7

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there is a δ-neighborhood $V_\delta(c)$ of c such that

if $x \in V_\delta(c)$, then $f(x) > \frac{f(c)}{2}$

Pf. Let $\varepsilon = \frac{f(c)}{2}$. Then there is a $\delta > 0$ such that

if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}$.

Hence,

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}.$$

Adding $f(c)$ to both sides of the left inequalit, we obtain

$\frac{f(c)}{2} < f(x)$. This is a stronger version of #7.

3. #11. Let $K > 0$ and let $f: \mathbb{R} \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leq K|x-y|,$$

for $x, y \in \mathbb{R}$.

Pf. Let $\varepsilon > 0$ and set $\delta = \dfrac{\varepsilon}{K}$.

Let $y = c$. If $|x - c| < \delta$,

then $|f(x) - f(c)| \leq K|x - c|$

$$< K\delta = K \cdot \frac{\varepsilon}{K} = \varepsilon. \checkmark$$

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that

$f(x) = 0$ for all $x \in \mathbb{R}$.

Pf. We want to show that $f(s) = 0$ for every irrational number s. For this purpose, let r_n be a rational number th $s < r_n < s + \frac{1}{n}$. (This is possible since the rationals are dense. Then (r_n) converges to r.)

Since f is continuous at s,

it follows that

$$\lim f(r_n) = f(s).$$

Since $f(r_n) = 0$ for all n,

$f(s) = 0$. Since s is

irration, it follows that

$f(x) = 0$ for all x. ✓

5.1.3 p. 126 Sequential Criterion for Continuity.

Suppose that $f: A \to \mathbb{R}$ is continuous at a given point $c \in A$.

Then for every sequence (x_n) with $\lim(x_n) = c$, $x_n \neq c$, it must be that

$\lim(f(x_n)) = f(c)$.

Conversely, if there is just a single sequence with $\lim x_n = c$ and $x_n \neq c$, then $\lim (f(x_n)) \neq f(c)$.

p. 134 #7. Give an example of a function $f: [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$, but such that $|f|$ is continuous at every point of $[0,1]$.

Pf. Note that the Dirichlet function f takes on 2 values, 0 and 1. Set $F(x) = f(x) - \frac{1}{2}$.

Then F takes only

$1-\frac{1}{2}$ or $0-\frac{1}{2}$. In either

case, $|F(x)| = \frac{1}{2}$ for all

$x \in [0,1]$.

p. 140, # 13.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$\lim_{x \to} f = 0 \quad \text{and} \quad \lim_{x \to \infty} f$$

Show that f is bounded on $\mathbb{R}$ and that f attains either a minimum or a maximum. Show that both a minimum and a maximum need not be attained.