Some Problems about Uniform Continuity.

Thm. Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$, and if f and g are <u>both</u> bounded on A, fg is uniformly continuous on A.

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Assume $|f(x)| \leq M$ and

$|g(x)| \leq M$ for all $x \in A$.

For x and $y \in A$,

$$|f(x)g(x) - f(y)g(y)|$$

$$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

$$\leq |f(x)(g(x) - g(y))| + |(f(x) - f(y))||g(y)|$$

$\leq M|g(x)-g(y)| + M||f(x)-f(y)|$

Now choosing any $\varepsilon > 0$,

choose $\delta_1 > 0$, so that

$|g(x)-g(y)| < \frac{\varepsilon}{2M}$ when $|x-y| < \delta_1$ (*)

Similarly, choose $\delta_2 > 0$ so

that if $|x-y| < \delta_2$,$^{(*)}$ then

$|f(x)-f(y)| < \frac{\varepsilon}{2M}$.

(*) since both f and g

are uniformly continuous

The last two terms in (2)

satisfy

$$\le M\,|g(x) - g(y)| + M\,|f(x) - f(y)|$$

Since both f and g are

bounded above by

$$M \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} M$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

#9. If f is uniformly continuous on A and if $|f(x)| \geq m$ for all $x \in A$, show that f is uniformly continuous.

Choose any 2 points x and y in A.

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right|$$

$$= \frac{|f(y) - f(x)|}{|f(x)||f(y)|}$$

$$\leq \frac{|f(y) - f(x)|}{m^2}$$

Since f is uniformly continuous on A, for any $\varepsilon > 0$, there is $\delta > 0$ so that if $|x-y| < \delta$, then

$$|f(x)-f(y)| < m^2 \varepsilon$$

Hence, $\dfrac{|f(x)-f(y)|}{m^2} < \dfrac{m^2 \varepsilon}{m^2} = \varepsilon.$

Hence f is uniformly continuous.

#1, pg. 140.

Suppose that f is continuous a closed interval $[a, b]$, such that $f(x) > 0$ for all $x \in [a, b]$. Prove that there is a number $\alpha > 0$ so that $f(x) \geq \alpha$, for all $x \in [a, b]$.

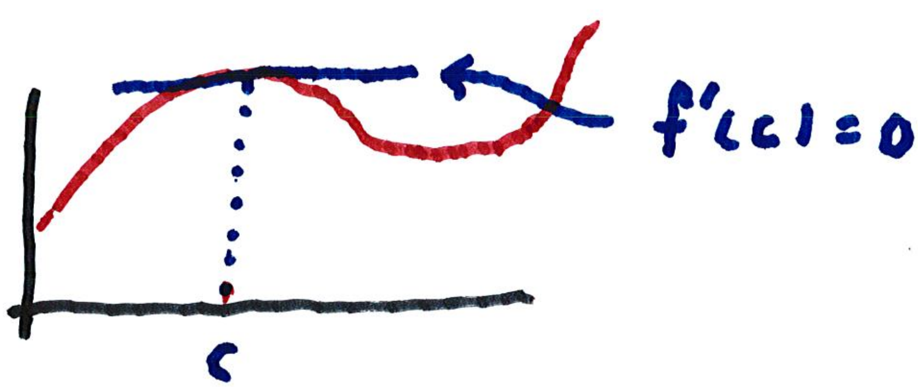
Define $g(x) = \frac{1}{f(x)}$ for all $x \in A$.

This function is continuous at each $x \in [a, b]$. Hence the Boundedness Theorem implies that there is number $M > 0$ so that $g(x) \leq M$.

$$\Rightarrow \frac{1}{f(x)} \leq M \Rightarrow f(x) \geq \frac{1}{M}$$

6.2 The Mean Value Theorem

Let $f: I \to \mathbb{R}$, where I is an interval. The function f has a relative maximum (or minimum) at $c \in I$ if there is a neighborhood $V_\varepsilon(c) = V$ of c such that $f(x) \le f(c)$ (or $f(x) \ge f(c)$) for all x in V.



Interior Extremum Theorem.

Let c be an interior point of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$

Pf. We prove the theorem in the case when f has a relative maximum.

If $f'(c) > 0$, then there is a neighborhood $V \subseteq I$ of c such that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ for all } x \in V,$$

with $x \neq c$.

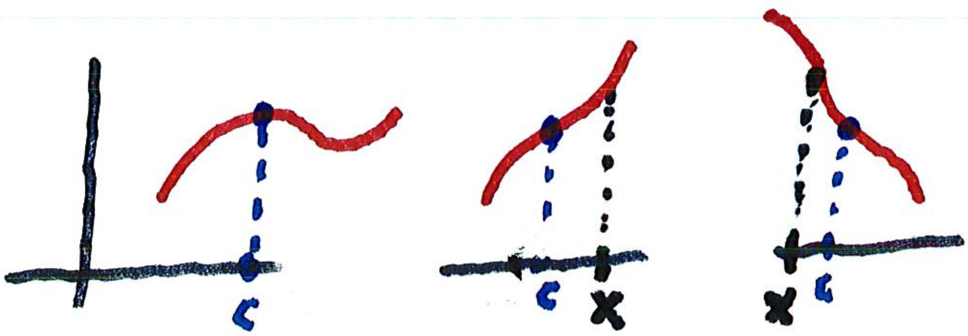
If $x \in V$ and $x > c$, then

$$f(x) - f(c) = (x - c)\,\frac{f(x) - f(c)}{x - c} > 0$$

This contradicts the hypothesis that f has a relative maximum at c.

Similarly, we cannot have

$f'(c) < 0$.



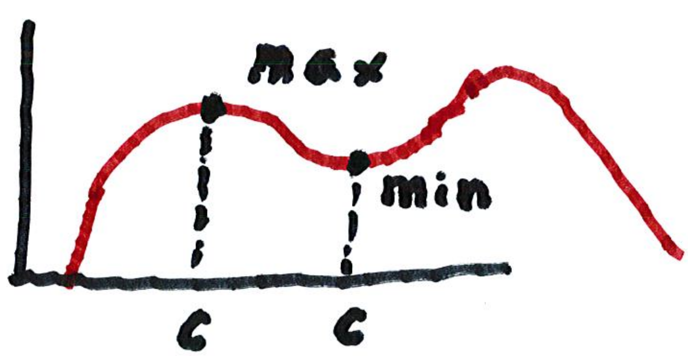
For if $f'(c) < 0$, then

$$\frac{f(x) - f(c)}{x - c} < 0, \quad \text{all } x \in V, \; x \neq c$$

If $x \in V$ and $x < c$, then

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

Rolle's Theorem. Suppose that $f: I \to \mathbb{R}$ is continuous on a closed interval $I = [a, b]$, that f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.



Proof. If $f(x) = 0$ for all c in (a, b), then any point c satisfies the conclusion of the theorem. Thus, we can assume that f does not vanish identically. Replacing f by $-f$ if necessary, we can assume that f assumes some positive values. By the

Maximum-Minimum Thm,

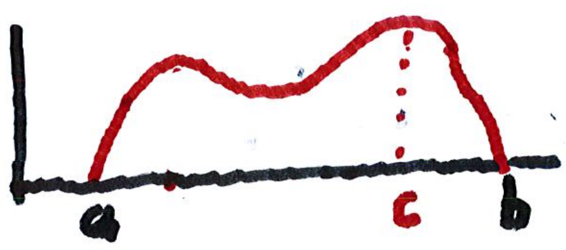
the function f attains the

value $\sup\{f(x) : x \in I\}$ at

some c in (a,b). Since

$f(a) = f(b) = 0$, the point

c must lie in (a,b).



Since

f has a relative maximum

at c, we conclude from

the Interior Extremum Theorem that $f'(c) = 0$.

We now prove the

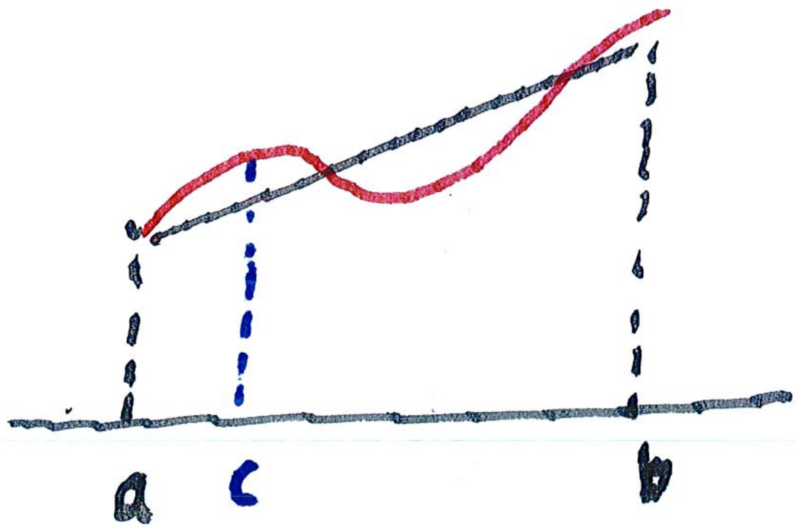
Mean Value Thm. Suppose that f is continuous on a closed interval $[a, b]$, and that f has a derivative in (a, b). Then there is a point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b-a)$$

Pf. Consider the function

$$\varphi(x) = f(x) - f(a) - \frac{f(b-a)}{b-a}(x-a).$$

(The function φ is the difference between f and the function whose graph is the line segment joining $(a, f(a))$ and $(b, f(b))$



Note that $\phi(a) = 0$ and $\phi(b) = 0$. We can apply Rolle's Thm, which implies that there is a point $c \in (a,b)$ such that $\phi'(c) = 0$. Hence

$$0 = \phi'(c) = f'(c) - \frac{f(b) - f(a)}{b-a}$$

It follows that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Thm. Suppose that f is continuous on $[a,b]$, that f is differentiable on (a,b) and that $f'(x) = 0$ for all $x \in (a,b)$. Then f is a constant on $[a,b]$

Pf. We will show that

$f(x) = f(a)$ for all $\cancel{x \in [a,b]}$.

$x \in [a,b]$. In fact,

if $x > a$, we apply the

Mean Value Theorem to

f on the closed interval

$[a,x]$. We obtain a

number c (dependent on x)

between a and x so that

$$f(x) - f(a) = f'(c)(x-a).$$

(by the MVT)

Since $f'(c) = 0$, we deduce

that $f(x) - f(a) = 0$.

Corollary: Suppose that

f and g are continuous

on $[a,b]$, that they are

differentiable on (a,b) and

that $f'(x) = g'(x)$, for all $x \in [a,b]$

Then there is a constant C so that $f = g + C$.

Pf. Just apply the above theorem to $f - g$.

We say that $f : I \to \mathbb{R}$ is increasing on I if whenever $x_1, x_2 \in A$ with $x_1 < x_2$, then $f(x_1) \leq f(x_2)$.

Also f is decreasing if $-f$ is increasing.

Thm. Let $f : I \to \mathbb{R}$ be differentiable on I. Then

(a) f is increasing if and only if $f'(x) \geq 0$, all $x \in I$.

Pf. (a) Suppose that $f'(x) \geq 0$ for all $x \in I$. If x_1, x_2 in I satisfy $x_1 < x_2$, then the Mean Value Thm (applied to f on $[x_1, x_2]$) implies that there is a point

$c \in (x_1, x_2)$ such that

$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$.

Since $f'(c) \geq 0$, we conclude that

$f(x_2) - f(x_1) \geq 0$. Hence

f is increasing on I.

Now assume that f is

increasing on I, and

differentiable on I. Then

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

Passing to the limit, we obtain that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

6.3 L'hospital's Rules

Suppose that f, g are functions defined near c and that

$$A = \lim_{x \to c} f(x)$$ and

$$B = \lim_{x \to c} g(x).$$

If $B \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

If $A = 0$ and $B = 0$, then

the situation is more

complicated. L'Hospital's

Rules handle this situation.

We will need a generalization of the Mean Value Theorem.

Cauchy Mean Value Theorem.

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Assume $g'(x) \neq 0$ for all x in (a,b). Then there is a point c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Pf. Note that the hypothesis $g'(c) \neq 0$ implies that $g(b) \neq g(a)$ (by Rolle's Thm). For x in $[a,b]$ we define

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x)-g(a)\big) - \big(f(b)-f(a)\big)$$

Then h is continuous on $[a,b]$,

differentiable on (a,b), and $h(a) = h(b) = 0$.

Therefore Rolle's Thm. implies that there is a point c in (a,b) such that

$$0 = h'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) - f'(c).$$

Since $g'(c) \neq 0$, we can divide by $g'(c)$ to obtain the desired result.