We show (again) that the following holds.

L'Hopital's Rule.

Suppose that there are two functions f and g that are defined and differentiable on (a, b).

Suppose that

(1) $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$.

and also that $g'(x) \neq 0$ in (a, b).

Finally, assume that

$$\lim \frac{f'(x)}{g'(x)} = L, \quad \text{where } L \in \mathbb{R}.$$

Then $$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Two remarks:

(i) $g(x) \neq 0$ in (a, b), for

if $g(x) = 0$, then ~~th~~ Rolle's

theorem would imply that

$g'(c) = 0$, for $c \in (a, b)$.

(ii). Since (i) holds, we can assume that f and g are both continuous and $= 0$ at a.

Pf. of L'Hopital's Rule:

By the Cauchy Mean Value, for each $x \in (a,b)$, there is an $\alpha_x \in (a, x)$ such that

$$\left[f(x) - \underset{\substack{\uparrow \\ f(a)}}{0} \right] g'(\alpha_x) = \left[g(x) - \underset{\substack{\uparrow \\ g(a)}}{0} \right] f'(\alpha_x).$$

or $$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)} . \qquad (2)$$

By assumption, $\lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$

Since $a < \alpha_x < x$, it follows that

$$\lim \text{ fim } \frac{f'(\alpha_x)}{g'(\alpha_x)} \to L, \text{ so}$$

that (2) implies $\lim_{x \to a} \frac{f(x)}{g(x)} = L$,

This proves the theorem

Ex. Compute $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

(This has the indeterminate form $0/0$)

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} \underset{L'H.}{=} \lim_{x \to 0} \frac{\sin x}{2x}.$$

which also has the indet./form $0/0$.

One more time this limit $\underset{L'H}{=} \frac{\cos x}{2}$

$= \frac{1}{2}$

Ex. Compute $\lim_{x \to \infty} (1 + a/x)^x$

We first take ln:

$$\ln\left[\left(1 + \frac{a}{x}\right)^x\right] = x \ln\left(1 + \frac{a}{x}\right)$$

This is not a quotient, so we

write it as

$$\lim_{x \to \infty} \left(\frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}} \right) = \cancel{\frac{\frac{1}{\left(1+\frac{a}{x}\right)^{-2}}}{-}}$$

$$= \lim_{x \to \infty} \frac{\dfrac{1}{\left(1 + \frac{a}{x}\right)} \cdot \dfrac{-a}{x^2}}{-\dfrac{1}{x^2}}$$

L'H.

$$= \lim_{x \to \infty} \frac{a}{\left(1 + \frac{a}{x}\right)} = \frac{a}{1} = a.$$

We have shown that

$$\lim \ln \left(1 + \frac{a}{x}\right)^x = a$$

If we apply the exponential map (which is continuous at a) we get that

$$e^{\ln(1+\frac{a}{x})^x} \to e^a$$

or: $\lim (1+\frac{a}{x})^x = e^a$.

We now state and prove ~~that~~

Taylor's Theorem:

Let $n \in \mathbb{N}$, let $I = [a, b]$,

and let $f: I \to \mathbb{R}$ such that

f and its derivatives

$f', f'', \ldots, f^{(n)}$ are continuous

on I and that $f^{(n+1)}$ exists

on (a, b). If $x_0 \in I$, then

for any $x \in I$, there is a point c between x_0 and x such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n +$$

$$+ \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$