Section 7.4

The Darboux Integral

Suppose that f is a <u>bounded</u> function on $[a,b]$. Let

$$P = (x_0, x_1, \ldots, x_{n-1}, x_n)$$

be a partition of $[a,b] = I$.

Thus $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$.

For $k = 1, 2, \ldots, n$, we let

$$m_k = \inf \{ f(x) : x \in [x_{k-1}, x_k] \}$$

and $M_k = \sup \left\{ f(x) : x \in [x_{k-1}, x_k] \right\}$

The lower sum of (f, P) is

$$L(f;P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

and the upper sum is

$$U(f;P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

For a positive function,

$L(f,P) =$ sum of areas of rectangles with base $[x_{k-1}, x_k]$ and height m_k.

For an upper sum, the height is M_k.

Lemma 1. For any partition P and any f on $[a,b]$,

$$L(f,P) \leq U(f,P).$$

Pf. For any bounded set S.

$\inf S \leq \sup S$. $\therefore m_k = \inf\{f(x);\ x \in I_k\}$

Also $M_k = \sup\{f(x);\ x \in I_k\}$

Hence

$m_k \leq M_k$

If $P = \{x_0, x_1, \ldots, x_n\}$

and $Q = \{y_0, y_1, \ldots y_n\}$,

then we say Q is a refinement

of P, each element x_k belongs

to Q, i.e., $P \subset Q$. Hence

$[x_{k-1}, x_k] = [y_{j-1}, y_j] \cup [y_j, y_{j+1}] \cup$

$\ldots \cup [y_{n-1}, y_n]$

Lemma 2 If $f: I \longrightarrow \mathbb{R}$ is bounded,

if $\mathcal{P}$ is a partition of $I = [a, b]$,

and if $\mathcal{Q}$ is a refinement of $\mathcal{P}$,

then

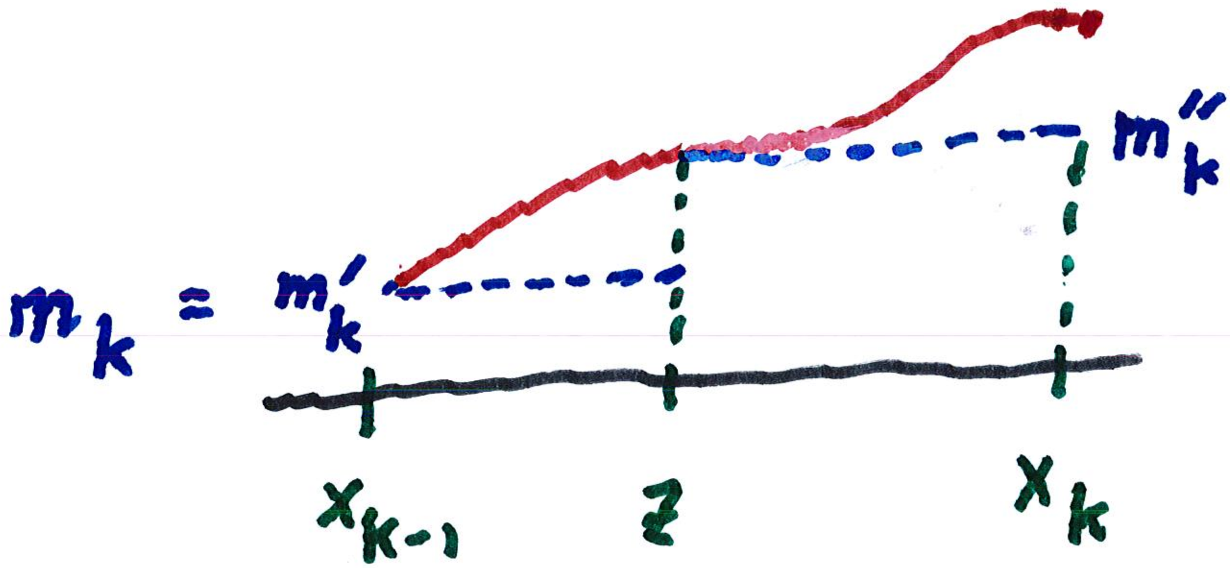
$L(f; P) \le L(f, Q)$ and

$U(f; Q) \le U(f; P)$.

Pf. Let $P = (x_0, x_1, \ldots, x_n)$.

First we assume that Q has

only one additional element


$m_k = m'_k$
m''_k
$x_{(k-1)}$
z
x_k

$z \in I$ satisfying

$$P' = (x_0, \dots, x_{k-1}, z, x_k, \dots, x_n)$$

Then define

$$m_k' = \inf \{ f(x) ; x \in [x_{k-1}, z] \} \text{ and}$$

$$m_k'' = \inf \{ f(x) ; x \in [z, x_k] \}.$$

Then $m_k \leq m_k'$ and $m_k \leq m_k''$.

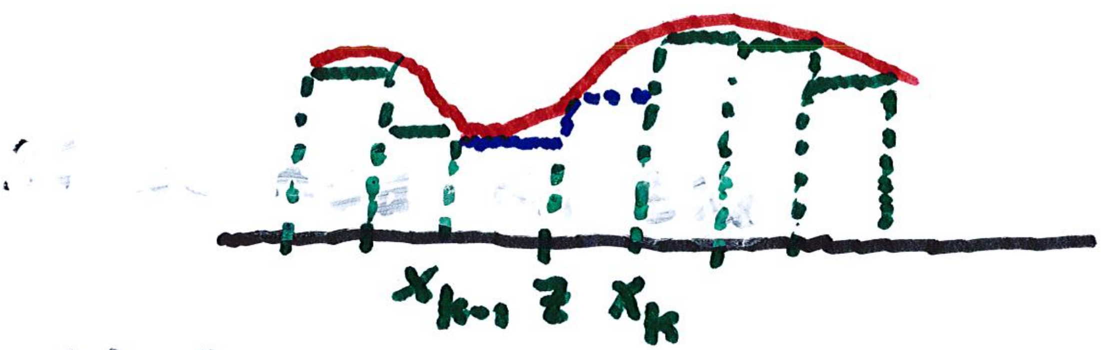
Hence

$$m_k (x_k - x_{k-1}) = m_k (z - x_{k-1}) + m_k (x_k - z)$$

$$\leq m_k' (z - x_{k-1}) + m_k'' (x_k - z).$$

If we add the terms $m_j(x_j - x_{j-1})$ for $j \neq k$, we obtain

$$L(f:P) \leq L(f,P').$$



If Q is obtained from P by adding a finite number of elements of Q, one at a time, then we obtain

$$L(f:P) \leq L(f;Q) \qquad U(f;Q) \leq U(f;P)$$

(Upper sums are handled similarly)

Lemma 3. Let $f: I \to \mathbb{R}$ be bounded. If P_1 and P_2 are any two partitions, then

$$L(f; P_1) \leq U(f, P_2).$$

Pf. Let $Q = P_1 \cup P_2$, then Q is a refinement of P_1 and P_2. Hence

Lemma 1 and Lemma 2 imply that

$$L(f: P_1) \le L(f: Q) \le U(f: Q) \le U(f: P_2)$$

Lemma 3 determines two sets of numbers.

$$\{L(f, P): P \in \mathcal{I}\}$$ and

$$\{U(f, P); P \in \mathcal{I}\}$$

Def'n. Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be bounded. The lower integral of f on I is the number

$$L(f) = \sup\{L(f;P) ; P \in \mathcal{P}(I)\}$$

and the upper integral of f on I is defined by

$$U(f) = \inf\{U(f;P) ; P \in \mathcal{P}(I)\}$$

Since f is a bounded function,

we have that

$$m_I = \inf\{f(x);\ x \in I\} \quad \text{and}$$

$$M_I = \sup\{f(x):\ x \in I\}$$

are both well-defined. In fact,

for any $P \in \mathcal{P}(I)$,

$$m_I(b-a) \leq L(f;P)$$

$$\leq U(f,P) \leq M_I(b-a).$$

Thm. Let $I = [a, b]$ and let $f: I \to \mathbb{R}$ be a bounded fcn.

Then the lower integral $L(f)$ and the upper integral $U(f)$ exist. Moreover

$$L(f) \le U(f).$$

Pf. If P_1 and P_2 are any partitions of I, then it follows from Lemma 3 that

$$L(f : P_1) \le U(f; P_2).$$

Therefore, $U(f : P_2)$ is an upper bound for the set

$$\{L(f:P);\ P \in \mathcal{P}(I)\}$$

Hence $L(f)$, which is the supremum of this set, satisfies $L(f) \le U(f; P_2)$

Since P_2 is an arbitrary

partition of I, then

$L(f)$ is a lower bound of

the set $\{U(f;P) : P \in \mathcal{P}(I)\}$

Therefore, $U(f)$ satisfies

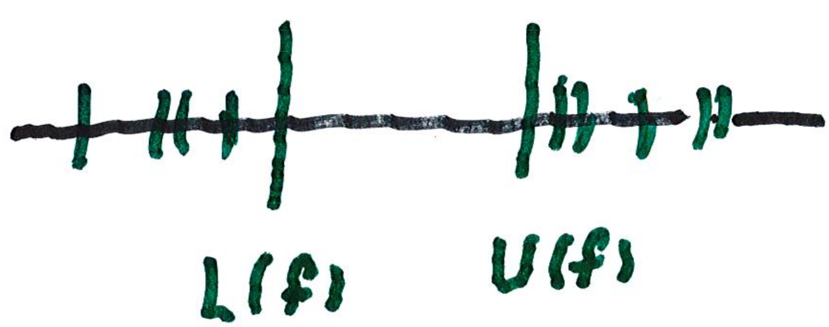
$L(f) \leq U(f)$.

Given any bounded function f on I, then

$$L(f) = \sup\{L(f,P) : P \subset I\}$$

and

$$U(f) = \inf\{U(f,P) : P \subset I\}$$



Def'n. Let $I = [a,b]$ and let $f: I \to \mathbb{R}$ be bounded. Then f is Darboux integrable on I if $L(f) = U(f)$. In this case, the Darboux integral of f over I is the value

$$L(f) = U(f).$$

Integrability Criterion.

Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. Then f is Darboux integrable on I if and only if for each $\varepsilon > 0$, there is a partition P_ε of I such that

$$U(f; P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Pf. If f is integrable, then

$L(f) = U(f)$. For a given $\varepsilon > 0$

there is a partition

P_2 of I such that

$$U(f; P_2) < U(f) + \frac{\varepsilon}{2}.$$

Similarly, there is a partition

P_1 of I so that

$$L(f, P_1) > L(f) - \frac{\varepsilon}{2}.$$

Now set $P = P_1 \cup P_2$. Then Lemma 1 and Lemma 2 imply that

$$L(f) - \frac{\varepsilon}{2} < L(f, P_2) \leq L(f, P)$$

$$\leq U(f, P) \leq U(f; P_2) < U(f) + \frac{\varepsilon}{2}$$

Since $L(f) = U(f)$, we conclude that $U(f:P) - L(f:P) < \varepsilon$