In this section, we will prove several identities that have to do with the absolute value function. But first we note that in the third line of the definition, it follows that

If $b < 0$, then $|b| = -b$.

and if $b \geq 0$, then $|b| = b$.

Absolute Value 2.2.

We can define $|a|$ as follows:

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases} \quad (*)$$

We'll need these identities:

(a) $|-a| = |a|$

(b) $|ab| = |a||b|$

(c) $|a|^2 = a^2$

(d) $-|a| \leq a \leq |a|$

(e) if $b < 0$, then $|b| = -b$.

Proof.

(a) Suppose $a \geq 0$. Then $-a \leq 0$

$\rightarrow |-a| = -(-a) = a = |a|$

If $a < 0$, then $-a > 0$, so

$|-a| = -a = |a|$

by def. of $|a|$
when $a < 0$

(b) If either a or $b = 0$, then

both sides equal 0.

Now suppose $a, b > 0$.

$|ab| = ab = |a||b|$

since $ab > 0$

Now suppose $a > 0$, $b < 0$.

$|ab| = -ab = a(-b) = |a||b|$

When $a < 0$ and $b > 0$, and

$a, b < 0$, the argument is

similar.

Now suppose that

$a < 0$ and $b < 0$. Then

$|ab| = |(-a)(-b)|$.

Since $-a$ and $-b$ are both > 0,

$|(-a)(-b)| = (-a)(-b)$

$= |-a||-b| = |a||b|$

by (a)

This proves (b).

Proof of (c).

Suppose first that $a \geq 0$.

Then $a = |a|$, so $a^2 = |a|^2$.

Now suppose that $a < 0$.

Then $-a = |a|$, so

$$a^2 = (-a)(-a) = |a||a|$$
$$= |a|^2.$$

This proves (c).

Proof of (d). We want to prove that

$-|a| \le a \le |b|$.

Suppose first $a \ge 0$.

Hence $a = |a|$

$\therefore -|a| \le 0 \le a = |a|$

Similarly, when $a < 0$,

$|a| = -a$. Then $-|a| = a \le 0 \le |a|$

which implies $-|a| \le a \le |a|$

This proves (d).

The following inequality is very useful.

Triangle Inequality.

If $a, b \in \mathbb{R}$, then

$|a+b| \le |a| + |b|$.

Pf. Suppose first that $a+b \ge 0$

$\rightarrow$ $|a+b| = a+b \le |a| + |b|$

using (d)

Now suppose that $a+b<0$

$\rightarrow |a+b| = -(a+b)$

$= -a-b \leq |a|+|b|$

using (d).

which implies the Triangle Inequality. We can prove

$|a-b| \leq |a|+|b| \qquad (1)$

by replacing b by $-b$.

We will also need:

$$\big||a|-|b|\big| \le |a-b| . \qquad (+)$$

pf.

$$a = (a-b) + b$$

$$|a| \le |a-b| + |b|$$

$$\rightarrow \ (|a|-|b|) \le |a-b| \qquad (2)$$

Similarly $b = b-a + a$

$$|b| \le |b-a| + |a|$$

$$|b| - |a| \leq |b - a|$$

$$-\left(|a| - |b|\right) \leq |a - b| \qquad (3)$$

By combining (2) and (3),

we obtain

$$\Big|\, |a| - |b| \,\Big| \leq |a - b|,$$

which proves (*).

Another version is the

Backwards Triangle Property

$$|a-b| \geq |a| - |b|.$$

Pf.

$$|a| = |(a-b) + b|$$

$$\leq |a-b| + |b|$$

$$\Rightarrow |a-b| \geq |a| - |b|$$

One more inequality:

<u>Estimate</u>. Suppose that $c \geq 0$.

(1) $|a| \leq c$ if and only if

$$-c \leq a \leq c.$$

Let P and Q be statements.

Then P is true if and only if

Q is true, means that

P is true if Q is true

i.e., $Q \Rightarrow P$

and

P is true only if Q is true.

i.e., $P \Rightarrow Q$.

We prove (1) in 2 separate cases

Case 1: Suppose $a \geq 0$.

Since $|a| \leq c$, $\Rightarrow$ $a \leq c$

$\rightarrow$ $-c \leq 0 \leq a \leq c$,

$\rightarrow$ $-c \leq a \leq c$.

On the other hand. if

$-c \leq a \leq c$. then $|a| \leq c$

Case 2. Suppose $a < 0$.

If $|a| \leq c$, then $-a \leq c$

$\rightarrow$ $a \geq -c$.

Hence, $-c \le a < 0 \le c$

$\rightarrow \quad -c \le a \le c.$

On the other hand, if

$-c \le a \le c$, then

$-a \le c \longrightarrow \quad |a| \le c.$

This proves (1) is true if

$a < 0.$

This proves the estimate.
in both cases.

We obtain $|a| \le c$

Thus, we've proved both directions.

Ex. Find the set A of all x such that $|3x+4| < 2$

$\therefore$ Left half is

Set $c = 2$ and $a = 3x+4$.

$|a| < c \to -c < a < c$

or $-2 < 3x+4 < 2$

$\therefore \quad -6 < 3x < -2$

$\rightarrow \quad -2 < x < -\frac{2}{3}.$

Ex. Set $f(x) = \dfrac{2x^2 - 4x + 3}{5x - 2}.$

when $1 \le x \le 2$. Estimate $|f(x)|$.

For the numerator;

$|2x^2 - 4x + 3| \le |2x^2| + |4x| + 3$

$\le 8 + 8 + 3 = 19$

For the denominator:

$$|5x-2| \geq |5x| - |2|$$

$$\geq 5 - 2 = 3$$

Hence,

$$|f(x)| \leq \frac{19}{3}$$

Def'n. Let $a \in \mathbb{R}$ and $\varepsilon > 0$.

Then the ε-neighborhood of a is the set

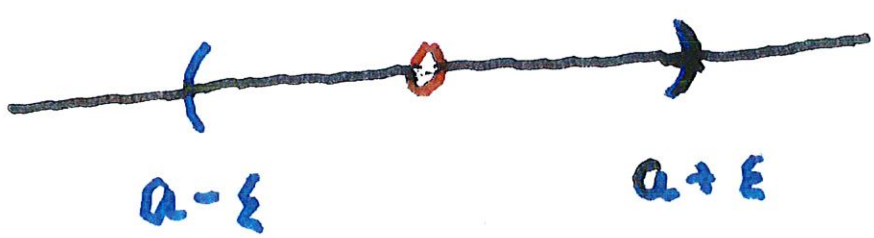
$$V_\varepsilon(a) = \{ x \in \mathbb{R} : |x-a| < \varepsilon \}.$$

If we replace a in (1) by $x-a$ and c by ε, it follows that $x \in V_\varepsilon(a)$ if only if

$$-\varepsilon < x-a < \varepsilon$$

or $a-\varepsilon < x < a+\varepsilon$

On the real line this is



Thm. Let $a \in \mathbb{R}$. If x belongs to $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Pf. Suppose $x \neq a$. If we set $\varepsilon = \frac{|x-a|}{2}$ in the definition of $V_\varepsilon(a)$, then

$$|x-a| < \frac{|x-a|}{2}.$$

Dividing by $|x-a|$, we have $1 < \frac{1}{2}$. This contradiction $\rightarrow$ $x = a$.