Answers for Exam 1, MA 341

1. $$\frac{n\left(2+\frac{3}{n}\right)}{n\sqrt{3+\frac{2}{n^2}}} = \frac{2+\frac{3}{n}}{\sqrt{3+\frac{2}{n^2}}} \to \frac{2+0}{\sqrt{3+0}} = \frac{2}{\sqrt{3}}$$

2a. Let $\varepsilon > 0$. Since (y_n) is bounded

$|y_n| \leq M$, for all n

$\lim\limits_{n\to\infty} (x_n) = 0 \to |x_n| < \frac{\varepsilon}{M}$, all n

$\therefore |x_n y_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$. $\therefore \lim x_n y_n = 0$

(b) $\lim 2x_n = 0$ and $\lim x_n y_n = 0$

$$\therefore \frac{2x_n + 3}{2 + x_n y_n} \to \frac{0+3}{2+0} = \frac{3}{2}$$

3. Let $\varepsilon = \frac{|z|}{2}$. There is an integer $K > 0$ so that if $n \geq K$, then $|z_n - z| < \frac{|z|}{2}$. Hence,

$$|z_n| = |z + (z_n - z)| \geq |z| - |z_n - z|$$

$$> |z| - \frac{|z|}{2} = \frac{|z|}{2}$$

4. By #3, there is $K_1 > 0$ so that if $n \geq K_1$, then $|z_n| > \frac{|z|}{2}$. Hence

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z - z_n|}{|z||z_n|} < \frac{2|z - z_n|}{|z|^2}$$

Since $\lim (z_n) = z$, there is $K_2 > 0$ so that $|z_n - z| < \frac{|z|^2 \varepsilon}{2}$. Let $K = \max\{K_1, K_2\}$. If $n \geq K$, then

$$\frac{2|z-z_n|}{|z|^2} < \frac{2|z|^2}{|z|^2}\frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim \frac{1}{z_n} = \frac{1}{z}$.

5a. Suppose $p = 3$. Then if $x_n < 3$, then

$$x_{n+1} < \frac{6}{5} + \frac{3}{2} = \frac{27}{10} < 3.$$

b. Note that $x_1 < x_2$. Assume $x_n < x_{n+1}$. Then

$$\frac{2}{5}x_n < \frac{2}{5}x_{n+1} \Rightarrow \frac{2}{5}x_n + \frac{3}{2} < \frac{2}{5}x_{n+1} + \frac{3}{2}.$$

Hence $x_{n+1} < x_{n+2}$.

(c). Since a. and b. imply that (x_n) is bounded and increasing, the Monotone Convergence Thm $\Rightarrow$ (x_n) has a limit $= x$.

(d) Since $\lim(x_n)$ and $\lim(x_{n+1})$, $= x$, it follows that $x = \frac{2}{5}x + \frac{3}{2}$

$\Rightarrow x = \frac{5}{2}$.

6a. A bounded sequence has a convergent subsequence

b. An upper bound u is a number u that satisfies $u \geq s$ for all $s \in S$.

c. u is a supremum of S if

(i) u is an upper bound of S and

(ii) If v is also an upper bound of S, then $v \geq u$.

7a. Suppose (x_n) is a sequence of positive numbers, ~~and~~ and ~~if~~ that $\lim (x_{n+1}/x_n) = L$. If $L < 1$, then $\lim x_n = 0$.

(b) $\lim \frac{2n+1}{3^{n+1}} \cdot \frac{3^n}{2n-1} = \lim \frac{(2n+1)}{3(2n-1)} = \frac{1}{3}$.

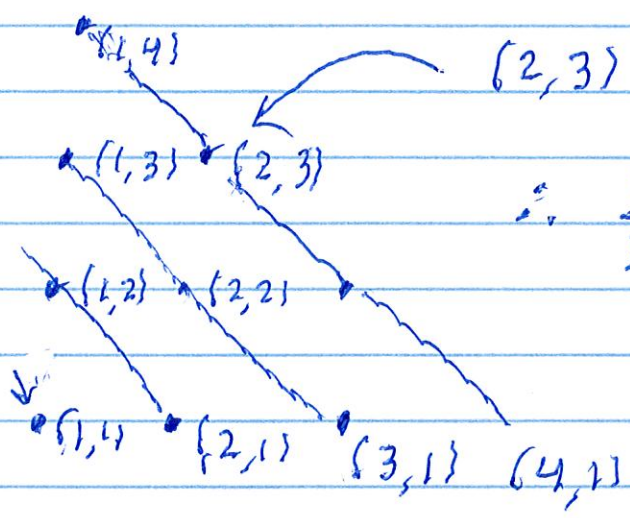
Hence $\lim \frac{2n-1}{3^n} = 0$.

8a. We want to find a bijection of $\mathbb{N}$ onto $\mathbb{Z}$

$$f(n) = [\![\tfrac{n}{2}]\!] (-1)^n,$$

where $[\![x]\!]$ = largest integer $\leq x$.

b



(2,3) = 7-th point

$\therefore \frac{2}{3}$ = 7-th rational

(skip (2,2)