Definition. We say a sequence (x_n) is increasing if

$$x_n \le x_{n+1}, \qquad \text{all } n = 1, 2, \ldots$$

That is

$$x_1 \le x_2 \le \ldots x_n \le x_{n+1} \le \ldots$$

We say (y_n) is decreasing if

$$y_n \ge y_{n+1}, \qquad n = 1, 2, \ldots$$

That is

$$y_1 \ge y_2 \ge \ldots \ge y_n \ge y_{n+1} \ge \ldots$$

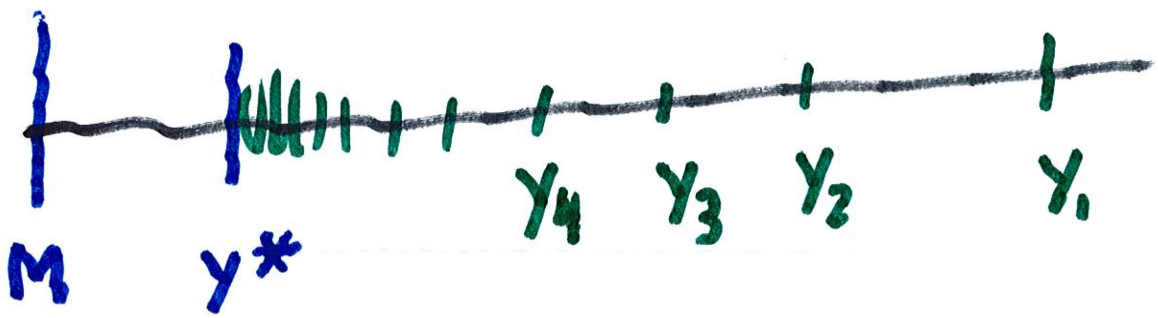
If (x_n) is increasing or decreasing, we say (x_n) is monotone.

Monotone Convergence Thm.

If (x_n) is a bounded monotone sequence, then it converges. In fact, if (x_n) is increasing and bounded, then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$

Also, if (y_n) is decreasing, then

$$\lim(y_n) = \inf\left\{ y_n : n \in \mathbb{N} \right\}.$$



Pf. when (x_n) is inc. and bounded.

.t $x^* = \sup\left\{ x_n : n \in \mathbb{N} \right\}$, then for any $\varepsilon > 0$, $x^* - \varepsilon$

is not an upper bound.

Hence there is an x_K so

that $x_K > x^* - \varepsilon$.

Since (x_n) is increasing,

$x_n \geq x_K$, all $n \geq x_K$.

Hence

$$x^* - \varepsilon < x_K \leq x_n \leq x^* < x^* + \varepsilon$$

It follows that if $n \geq K$, then $|x_n - x^*| < \varepsilon$.

Similarly, if (y_n) is decreasing and bounded, and if

$$y^* = \inf\{y_n ; n \in \mathbb{N}\},$$

then for any $\varepsilon > 0$,

$y^* + \varepsilon$ is not an upper bound of $\{y_n : n \in \mathbb{N}\}$.

Hence as before, there is element y_K so that

$$x_K > x^* - \varepsilon .$$

Again, since (y_n) is decreasing,

$$y_n \leq y_K , \quad \text{for all } n \geq K.$$

Hence,

$$y^* - \varepsilon < y^* \leq y_n \leq y_K < y^* + \varepsilon$$

Hence, if $n \geq K$, $|y_n - y^*| < \varepsilon$.

Ex. Define $y_{n+1} = \frac{2}{5} y_n + 1$,

with $y_1 = 1$.

Assume first

that $\lim y_n = y$. Then we have

$$y = \frac{2}{5} y + 1 \to \frac{3}{5} y = 1$$

$$\to \underline{y = \frac{5}{3}}.$$

Use induction to show that

if $1 \le y_n \le 4$, then y_{n+1}

satisfies $1 \le y_{n+1} \le 4$.

We must show that

$$y_{n+1} > y_n . \qquad (*)$$

When $n=1$, this is easy.

Suppose by induction that (*) holds. Then

$$\frac{2}{5} y_{n+1} > \frac{2}{5} y_n$$

$$\frac{2}{5} y_{n+1} + 1 > \frac{2}{5} y_n + 1$$

This gives

$$y_{n+2} > y_{n+1}.$$

It follows that

(y_n) is increasing.

Hence the Mon. Conv. Thm

$\rightarrow (y_n)$ converges to y

satisfying $1 \leq y \leq 4$.

As noted above $\lim y_n = \frac{5}{3}$

Calculation of Square Roots.

Let $a > 0$. We construct (s_n) that converges to $\sqrt{a}$.

{Known in Mesopotamia before 1500 B.C.}

Let $s_1 > 0$ be arbitrary and define $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$

Note first that $s_n > 0$ for all $n = 1, 2, \ldots$

We first show that

$$s_n^2 \geq a \quad , \qquad \text{all } n = 2, \ldots$$

Since s_n satisfies the quadratic equation

$$2 s_n s_{n+1} = s_n^2 + a.$$

or

$$(s_n - s_{n+1})^2 = s_{n+1}^2 - a .$$

Since the root s_n is real,

it follows that $s_{n+1}^2 - a \geq 0$

for all $n \geq 1$.

To see that (s_n) is ultimately decreasing, we note that for $n \geq 2$,

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$

$$= \frac{1}{2}\,\frac{(s_n^2 - a)}{s_n} \geq 0 .$$

Hence, $s_{n+1} \leq s_n$, for all $n \geq 2$.

The Monotone Convergence Thm. implies that $\lim (s_n)$ exists. It follows that

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right)$$

and so, $s = \frac{a}{s} \rightarrow \underline{\underline{s^2 = a}}$