3.6 Properly Divergent Series

Let (x_n) be a sequence.

(i) We say (x_n) tends to $+\infty$ and write $\lim(x_n) = +\infty$ if for every $\alpha \in \mathbb{R}$, there exists a nat. number $K(\alpha)$ such that if $n \geq K(\alpha)$, then $x_n > \alpha$.

(ii) We say (x_n) tends to $-\infty$

and write $\lim (x_n) = -\infty$

if for every $\beta \in \mathbb{R}$,

there exists ~~$K(\beta) \in \mathbb{N}$~~

a nat. number $K(\beta)$ such

that if $n \geq K(\beta)$, then

$x_n < \beta$.

In either case, we say (x_n)

is properly divergent.

Ex. $\lim(n) = +\infty$,

because if α is given, let $K(\alpha)$ be any natural number such that $K(\alpha) > \alpha$.

If $n \geq K(\alpha)$, then $n > \alpha$.

Ex. $\lim(n^2) = +\infty$ Because if $K(\alpha) > \alpha$, and if $n \geq K(\alpha)$ then $n^2 \geq n > \alpha$.

Ex. If $c > 1$, then $\lim c^n = +\infty$

In fact, let $c = 1 + b$. If α is given, let $K(\alpha)$ be a natural number such that $K(\alpha) > \frac{\alpha}{b}$. If $n \geq K(\alpha)$, it follows from Bernoulli's Inequality that

$$c^n = (1+b)^n \geq 1 + nb > 1 + \alpha > \alpha.$$

$\uparrow$

Note that the inequalities

$n \geq K(\alpha) > \frac{\alpha}{b}$ imply that

$$n > \frac{\alpha}{b} \longleftrightarrow nb > \alpha.$$

Recall that the Monotone Convergence Thm states that a monotone sequence is convergent if and only if it's bounded.

Similarly, we have:

Thm. A monotone sequence is properly divergent if and only if it is unbounded.

(a) If (x_n) is an unbounded increasing sequence, then $\lim(x_n) = +\infty$

(b) If (x_n) is an unbounded decreasing sequence, then $\lim(x_n) = -\infty$.

Comparison Test:

Thm. Let (x_n) and (y_n) be two sequences and suppose that $x_n \leq y_n$, all $n \in \mathbb{N}$

(a) If $\lim(x_n) = +\infty$,

then $\lim(y_n) = +\infty$

(b) If $\lim(y_n) = -\infty$, then

$\lim(x_n) = -\infty$

Ex. $\lim(\sqrt{n}) = +\infty$.

Let $K(\alpha)$ be any natural number with $K(\alpha) > \alpha^2$. If $n \geq K(\alpha)$, then $n > \alpha^2$. which implies $\sqrt{n} > \alpha$.

Compute $\lim\{\sqrt{n+2}\}$

Note that if we use the same $K(\alpha)$ as above, Then, if $n > \alpha^2$, then

$$\sqrt{n+2} > \sqrt{n} > \alpha.$$

which implies $\lim(\sqrt{n+2}) = +\infty$.

OR, we could have used the above convergence test, with $x_n = \sqrt{n}$ and $y_n = \sqrt{n+2}$.

Since $\lim(\sqrt{n}) = +\infty$, we get

$$\lim(\sqrt{n+2}) = +\infty.$$

Compute $\lim\left(\frac{\sqrt{n^2+1}}{\sqrt{n}}\right)$.

$$\underbrace{\frac{\sqrt{n^2+1}}{\sqrt{n}}}_{\text{Set } y_n} > \frac{\sqrt{n^2}}{\sqrt{n}} = \underbrace{\sqrt{n}}_{\text{Set } x_n}.$$

$\therefore$ Comp Test $\Rightarrow \lim\left(\frac{\sqrt{n^2+1}}{\sqrt{n}}\right) = +\infty$.

Ex. What about $\frac{\sqrt{n}}{(n^2+1)}$

Note that $\frac{\sqrt{n}}{(n^2+1)} < \frac{\sqrt{n}}{n^2} < \frac{n}{n^2}$

$= \frac{1}{n}.$

Since $\lim \frac{1}{n} = 0$,

so does $\lim \frac{\sqrt{n}}{(n^2+1)} = 0$

Limit Comparison Test.

Suppose (x_n) and (y_n)

are positive, and that

$\lim \left\{ x_n / y_n \right\} = L \neq 0.$

Then $\lim x_n = +\infty$

if and only if $\lim y_n = +\infty$

Set $x_n = \dfrac{\sqrt{2n^2+1}}{\sqrt{3n-1}}$

and $y_n = \dfrac{n}{\sqrt{n}} = \sqrt{n}.$

Check $\lim \dfrac{x_n}{y_n}.$

$$\frac{x_n}{y_n} = \frac{\dfrac{\sqrt{2n^2+1}}{\sqrt{3n-1}}}{\sqrt{n}}$$

$$= \frac{\sqrt{2n^2+1}}{\sqrt{n}\cdot\sqrt{3n-1}}$$

$$= \frac{n\sqrt{2+1/n^2}}{\sqrt{n}\cdot\sqrt{n}\cdot\sqrt{3-\frac{1}{n}}}$$

$$= \sqrt{\frac{2+1/n^2}{3-\frac{1}{n}}} \to \sqrt{\frac{2}{3}}$$

Proof of Limit Comparison Test.

We have $\lim \dfrac{x_n}{y_n} = L > 0.$

Set $\varepsilon = \dfrac{L}{2}.$

$\rightarrow \quad L - \dfrac{L}{2} < \dfrac{x_n}{y_n} < L + \dfrac{L}{2}$

if n is $> K_1$.

$\rightarrow \quad \dfrac{L}{2} < \dfrac{x_n}{y_n} < \dfrac{3L}{2}$

or $\dfrac{L}{2} y_n < x_n < \dfrac{3L}{2} y_n$

Hence the usual Comparison Test implies:

If $\lim y_n = +\infty$, then

$$\lim x_n = +\infty.$$

and if

$\lim x_n = +\infty$, then

$$\lim y_n = +\infty.$$

Compute $\lim (3n)^{1/2n}$

(We use $\lim n^{1/n} = 1$ at end of 3.1.)

$= \lim 3^{1/2n} \cdot n^{1/2n}$

$= \lim \left(3^{1/n}\right)^{\frac{1}{2}} \cdot \left(n^{\frac{1}{n}}\right)^{\frac{1}{2}}$

Note that $1 \le 3 \le n$

$\therefore \quad 1^{\frac{1}{n}} \le 3^{\frac{1}{n}} \le n^{\frac{1}{2n}}$

$\downarrow$

conv. to 1

$\therefore \left(3^{1/n}\right)^{\frac{1}{2}} \to \sqrt{1}$

Also, $n^{\frac{1}{n}} \to 1$,

so $\left(n^{\frac{1}{n}}\right)^{\frac{1}{2}} \to \sqrt{1}$.

Compute $\lim \left(1+\frac{1}{2n}\right)^{3n}$

$$= \lim \left(\underbrace{\left(1+\frac{1}{2n}\right)^{2n}}_{\text{conv. to } e}\right)^{3/2}$$

Because this a subsequence

of $\left(1+\frac{1}{k}\right)^k$.