3.7 Infinite Series

To define an infinite series of the form $\sum_{n=1}^{\infty} x_n$,

we define a sequence

$$S_N = \sum_{n=1}^{N} x_n \quad \text{for } N = 1, 2, \ldots$$

If the sequence S_N converges to S, we say the series $\underbrace{\text{and}}_{}$ converges and we write $\sum_{n=1}^{\infty} x_n = S$.

Ex. Consider the series

$\sum_{n=0}^{\infty} r^n$. If $r \neq 0$, then

$$S_N = \sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1-r}.$$

When $|r| < 1$, S_N converges

to $\frac{1}{1-r}$. Hence

$$\sum_{n=0} r^n = \frac{1}{1-r}.$$

Telescoping Series.

Ex. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its value.

Note that $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\therefore S_N = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots$$

$$+ \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

By cancellation:

$$S_N = \frac{1}{1} - \frac{1}{N+1} \longrightarrow 1 \quad \text{as } N \to \infty.$$

Suppose $\sum x_n$ converges.

Since $S_N \to S$ as $N \to \infty$,

given $\varepsilon > 0$, there is a K,

so that if $k \geq K$, then

$|S_k - S| < \varepsilon$.

But if $N \geq K+1$, then $N-1 \geq K$,

so $|S_{N-1} - S| < \varepsilon$.

Hence S_N and S_{N-1} both

converge to S.

If we write $S_N - S_{N-1} = x_N$,

then by letting $N \to \infty$, we

get $S - S = \lim_{N \to \infty} x_N$.

It follows that if $\sum_{n=1}^{\infty} x_n$,

then $\underline{\lim x_n = 0}$

Does $\sum_{n=1}^{\infty} \frac{\sqrt{2n^2-1}}{3n+5}$ converge?

Compute $\lim \frac{\sqrt{2n^2-1}}{3n+5}$

$$= \frac{n\sqrt{2-\frac{1}{n^2}}}{n\left(3+\frac{5}{n}\right)} \to \frac{\sqrt{2}}{3} \neq 0$$

as $n \to \infty$

Since (x_n) does NOT approach 0, it follows that the series diverges.

Ex. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Look at

$$S_{2^k} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$+ \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$\vdots$$

$$+ \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{k-1}}{2^k}$$

$$= 1 + \frac{k}{2} \longrightarrow \infty \text{ as } k \to \infty.$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Ex. For $p>1$, we want to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

We modify the above method:

$$S_{2^{k+1}-1} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots \frac{1}{7^p}\right)$$

$$+ \cdots + \left(\frac{1}{2^{kp}} + \frac{1}{(2^k+1)^p} + \cdots \frac{1}{(2^{k+1}-1)^p}\right)$$

$$S_{2^{k+1}-1} \leq 1 + \frac{2}{2^p} + \frac{4}{4^p} \cdots + \frac{2^k}{2^{kp}}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3$$

$$\cdots + \left(\frac{1}{2^{p-1}}\right)^k$$

If we set $r = \frac{1}{2^{p-1}}$, then

$$S_{2^{k+1}-1} = 1 + r + r^2 + \cdots + r^k.$$

$$= \frac{1 - r^{k+1}}{1 - r} < \frac{1}{1-r}.$$

If $n \geq 2^{k+1} - 1$, then

$S_n < \frac{1}{1-r}$. It follows

that S_n ~~is bounded~~

converges to a limit $\leq \frac{1}{1-r}$.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

for any $p > 1$.

Ex. If $p \leq 1$, then

$\frac{1}{n^p} \geq \frac{1}{n}$. Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

The last conclusion actually follows from the following:

Comparison Test. Suppose that (x_n) and (y_n) satisfy

$$0 \le x_n \le y_n, \qquad n \ge K.$$

Then

(a) The convergence of $\sum y_n$ implies the convergence of $\sum x_n$

(b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

For (a). Let S_n be the partial sum of $\sum x_n$ and let T_n be the partial sum of $\sum y_n$.

Clearly $S_n \leq T_n$. Since T_n is bounded for all n, it follows that $\sum x_n \leq \sum y_n$.

Ex. Determine the convergence

of $$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2-1}}{3n^3+4}$$

The n-th term is $\sim \dfrac{n}{n^3}$.

But if the denominator were

$3n^3+4$, we could use the

usual comparison test.

$$\frac{\sqrt{2n^2-1}}{3n^3+4} \le \frac{\sqrt{2n^2}}{3n^3} = \frac{\sqrt{2}}{3}\frac{1}{n^2}$$

It's better to use the Limit Comparison Test.

Suppose (x_n) and (y_n) are both positive and satisfy

$$r = \lim \left(x_n / y_n \right) \neq 0 .$$

Then $\sum x_n$ converges if and only if $\sum y_n$ converges.

Proof $\varepsilon = \frac{\pi}{2}$. Then there is a whole number K so that if $n \geq K$, then

$$\pi - \varepsilon < \frac{x_n}{y_n} < \pi + \varepsilon .$$

or
$$\frac{\pi}{2} < \frac{x_n}{y_n} < \frac{3\pi}{2} .$$

$$\left.\begin{array}{l} \text{Then } x_n < \frac{3\pi}{2} y_n \\ \text{and } y_n < \frac{2}{\pi} x_n \end{array}\right\} \begin{array}{c} \text{conv. of one} \\ \Rightarrow \text{conv. of other} \end{array}$$

$$\text{For } \sum \frac{\sqrt{2n^2-1}}{3n^3-4} \leftarrow x_n,$$

$$\text{Set } y_n = \frac{\sqrt{n^2}}{n^3} = \frac{1}{n^2}.$$

Must show

$$\lim \frac{\frac{\sqrt{2n^2-1}}{3n^3-4}}{\frac{1}{n^2}} = \frac{n^2 \cdot n\sqrt{2-1/n^2}}{n^3(3-4/n^3)}$$

$$\to \frac{\sqrt{2}}{3} \text{ as } n \to \infty. \text{ Since}$$

$\sum \frac{1}{n^2}$ conv., so does $\sum x_n$

The Limit Comp. Test does not apply to $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)}$.

There's no way to simplify x_n.

The integral test is best here.

$$\int_3^{\infty} \frac{1}{x \ln x} dx = \ln(\ln x) \Big|_3^{\infty} = \infty - \ln(\ln 3)$$

Also L'Hopital's Rule works, but we'll learn about these later.