4.2 Limit Theorems

Def'n. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let c be a cluster point of A. We say f is bounded on a neighborhood of c if there is a δ-neighborhood of c and a constant $M > 0$ such that

$|f(x)| \leq M$, for all x in $A \cap V_\delta(c)$.

Thm. If $A \subseteq \mathbb{R}$ and f has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.

Pf. If $\lim f = L$, then for $\varepsilon = 1$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$. Hence,

$$|f(x)| = |(f(x) - L) + L|$$

$$\leq |f(x) - L| + |L|$$

$$< 1 + |L|.$$

Thus, f is bounded on $V_\delta(c)$.

Thm. Let $A \subseteq \mathbb{R}$ and let f and g be functions on A. Suppose

$\lim\limits_{x \to c} f = L$ and $\lim\limits_{x \to c} g = M$.

Thm 1. $\lim\limits_{x \to c} (f+g) = L + M$,

2. $\lim\limits_{x \to c} (fg) = LM$

3 $\lim\limits_{x \to c} bf = bL$

4. If $h : A \to \mathbb{R}$ and $h(x) \neq 0$

and if $\lim\limits_{x \to c} h = H \neq 0$, then

$$\lim_{x \to c} \left(\frac{f}{h} \right) = \frac{L}{H}.$$

Pf. of 1. Choose $\delta_1 > 0$ so that

if $0 < |x-c| < \delta_1$, then

$|f(x) - L| < \frac{\varepsilon}{2}$.

Choose $\delta_2 > 0$ so that if

$0 < |x-c| < \delta_2$, then

$|g(x) - M| < \frac{\varepsilon}{2}$.

Now set $\delta = \min(\delta_1, \delta_2)$

If $0 < |x-c| < \delta$, then

$$\left|\{f(x)+g(x)\}-(L+M)\right|$$

$$=\left|(f(x)-L)+(g(x)-M)\right|$$

$$\leq \left|f(x)-L\right| + \left|g(x)-M\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves 1.

Pf. of 2., we have

$$\left|f(x)g(x)-LM\right|$$

$$= \left| \left(f(x)g(x) - Lg(x) \right) + \left(Lg(x) - LM \right) \right|$$

$$\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|$$

The above theorem shows that there is a $\delta_1 > 0$ so that if $0 < |x - c| < \delta_1$ and $x \in A$, then

1. $|g(x)| < M$ $\quad (M > 0)$

Let $\varepsilon > 0$. Then there is

$\delta_2 > 0$ so that if $0 < |x-c| < \delta_2$

so that $|f(x) - L| < \frac{\varepsilon}{2M}$ 2.

Similarly, there is $\delta_3 > 0$

so that if $0 < |x-c| < \delta_3$,

then $|g(x) - M| < \frac{\varepsilon}{2|L|+1}$. 3.

Now set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

If $0 < |x-c| < \delta$, then

all 3 inequalities 1., 2., 3

hold. Hence

$$|f(x)-L||g(x)| + |L||g(x)-M|$$

$$< \frac{\varepsilon}{2M} \cdot |M| + |L| \cdot \frac{\varepsilon}{2|L|+1}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since

$$\frac{|L|\,\varepsilon}{2|L|+1} < \frac{\varepsilon}{2}$$

This proves 2.

3. Also follows since we can just let $g(x) = b_0$ all x.

Proof of 4. By 3., it suffices to prove 4. with $f(x) = 1$, all x.

By the above boundedness theorem, there is $\delta_1 > 0$ so that if $0 < |x - c| < \delta_1$, then

4. $|h(x)| > \frac{H}{2}$ Hence,

$$\left| \frac{1}{h(x)} - \frac{1}{H} \right| = \left| \frac{h(x) - H}{h(x)\, H} \right|$$

$$< \frac{|h(x) - H| \cdot 2}{|H|^2}$$

Now let $\varepsilon > 0$. Choose $\delta_2 > 0$

so that if $0 < |x - c| < \delta_2$ then

$$|h(x) - H| < \frac{\varepsilon |H|^2}{2}. \qquad 5.$$

Now set $\delta = \min\{\delta_1, \delta_2\}$.

If $0 < |x-c| < \delta$, then

both 4. and 5. hold.

Hence

$$\frac{|h(x) - H| \cdot 2}{H^2} < \frac{\varepsilon H^2 \cdot 2}{2H^2} = \varepsilon$$

Ex. If $c \neq 0$, then by setting

$h(x) = \frac{1}{x}$ and $H = \frac{1}{c}$, then

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}.$$

Ex. Find $\lim_{x \to 2} \left(\dfrac{x^3 + 4x}{3x^2 - x - 2} \right)$

Note that if $q(x) = 3x^2 - x - 2$,

then $q(2) = 3 \cdot 4 - 2 - 2 = 8 \neq 0$

$\therefore$ by repeated applying the above limit laws, we get

$$\lim_{x \to 2} \left(\frac{x^3 + 4x}{3x^2 - x - 2} \right)$$

$$= \frac{\lim_{x \to 2} (x^3 + 4x)}{\lim_{x \to 2} (3x^2 - x - 2)} = \frac{8 + 8}{8} = 2$$

Ex. If $L_k = \lim_{x \to c} f_k$, then

one can show

$$\lim_{x \to c} (f_1 + \cdots + f_n)$$

$$= \lim_{x \to c} f_1 + \ldots + \lim_{x \to c} f_n$$

$$= L_1 + \cdots + L_n$$

Similarly, if $L = \lim_{x \to c} f$,

then $\lim_{x \to c} (f(x))^n = L^n$.

If $p(x) = a_n x^n + \cdots + a_0$.

then $\lim\limits_{x \to c} p(x) = a_n c^n + \cdots + a_0$

$= p(c)$.

If p and q are polynomials.

and $q(c) \neq 0$. then

$$\lim_{x \to c} p/q = \frac{p(c)}{q(c)}$$

Here are some analogous results:

Thm. Let $A \subseteq \mathbb{R}$, and let $f: A \to \mathbb{R}$. If $\lim_{x \to c} f$ exists, and if $a \leq f(x) \leq b$, for all $x \in A$, $x \neq c$, then $a \leq \lim_{x \to c} f \leq b$.

Pf. Suppose $L = \lim_{x \to c} f < a$

Set $\varepsilon = a - L > 0$.

Since $\lim\limits_{x \to c} f = L$, there is $\delta > 0$

so that if $0 < |x - c| < \delta$, then

$$|f(x) - L| < a - L, \quad \text{i.e.,}$$

$$-(a-L) < f(x) - L < a - L$$

$$\to f(x) < a. \quad \text{Contradiction.}$$

$\therefore \lim\limits_{x \to c} f \geq a$.

Similar when $L > b$.

Set $\varepsilon = L - b$.

Squeeze Thm. Let $A \subseteq \mathbb{R}$ and let $f, g, h : A \to \mathbb{R}$, with $f(x) \le g(x) \le h(x)$.

If $\lim\limits_{x \to c} f = L$ and $\lim\limits_{x \to c} g = L$, then $\lim\limits_{x \to c} g(x) = L$.

Pf. It follows from the convergence of f and h to L, that there is $\delta > 0$ so that if

$0 < |x-c| < \delta$, then

$|f(x) - L| < \varepsilon$ and $|g(x) - L| < \varepsilon$

Hence,

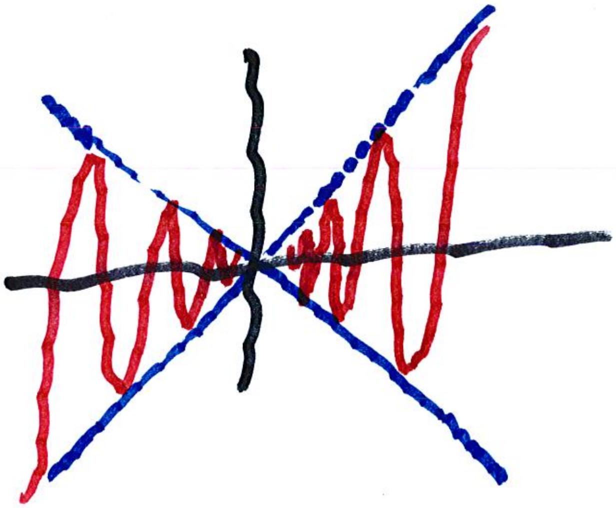
$f(x) - L \leq g(x) - L \leq h(x) - L$

and so:

$-\varepsilon < f(x) - L$ and $h(x) - L < \varepsilon$.

$\therefore \quad |g(x) - L| < \varepsilon$

Ex. Set $f(x) = x\sin(1/x)$, $x \neq 0$



Note that

$$-|x| \leq x\sin(1/x) \leq |x|$$

Since $|x|$ and $-|x|$

both have the limit $= 0$ at $c = 0$,

it follows that $\lim (x\sin(1/x)) = 0$