Continuity of Sequences

Remember we showed that if

$\lim_{x \to c} f = L$ and if $\lim x_n = c$ and $x_n \neq c$,

then $\lim (f(x_n)) = L$.

If $\lim_{x \to c} f = L$, then for all $\varepsilon > 0$,

there is $\delta > 0$ so that if $0 < |x - c| < \delta$,

then $|f(x) - L| < \varepsilon$.

Moreover, if $\lim(x_n) = c$, $\underset{x_n \neq c}{\text{with}}$,

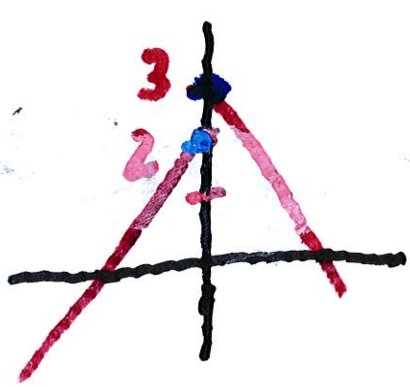
then there is $K > 0$ so that

if $n \geq K$, then $|x_n - c| < \delta$.

This means $|f(x_n) - L| < \varepsilon$,

so $\lim f(x_n) = L$

Ex. Define $f(x) = \begin{cases} 2+x, & \text{if } x \leq 0 \\ 3-x, & \text{if } x > 0. \end{cases}$



If $\lim f = L$,

then set $x_n = \frac{(-1)^n}{n}$,

If n is even, then

$$f(x_n) = 3 - \left(\frac{(-1)^n}{n} \right) = 3 - \frac{1}{n}.$$

$\rightarrow \lim_{\text{even}} f(x_n) = 3$

If n is odd, then

$$f(x_n) = 2 + \left(-\frac{1}{n}\right) = 2 - \frac{1}{n}.$$

and $\lim_{\text{odd } n} f(x_n) = 2$

$\therefore \lim f(x_n)$

does not exist.

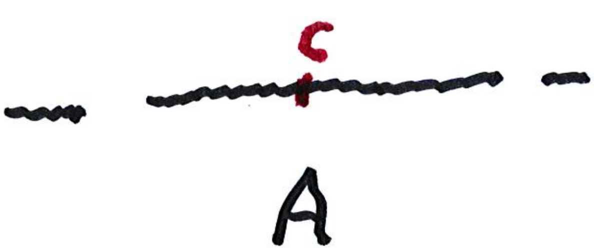
One-Sided Limits

We have a function f defined on a set A. Suppose that c is a cluster point of $A \cap (c, \infty)$. We say that L is a right-hand limit of f at c if for any $\varepsilon > 0$, there is

a $\delta > 0$ such that for all

$x \in A$ with $0 < x - c < \delta$,

then $\lim\limits_{x \to c^+} |f(x) - L| < \varepsilon$



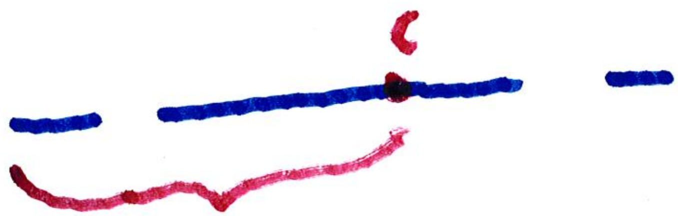
We only consider the function f on $A \cap (c, \infty)$

Similarly for the left hand limit, we assume c is a cluster point of $A \cap (-\infty, c)$

Then we say L is a left hand limit and we write

$\lim_{x \to c^-} f(x) = L$ or $\lim_{x \to c^-} f$

if given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$



$A \cap (-\infty, c)$

If c is a cluster point of $A \cap (c, \infty)$ and $A \cap (-\infty, c)$,

Then $\lim_{x \to c} f = L$ if and only if

$\lim_{x \to c^+} f = L$ and $\lim_{x \to c^-} f = L$.

Ex. Consider the function

$f(x) = \frac{1}{x^2}$. We want to

say that $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$

or more generally $\lim g(x) = +\infty$

or $\lim g(x) = -\infty$. as $x \to c$

Def'n. Let $A \subseteq \mathbb{R}$ and let

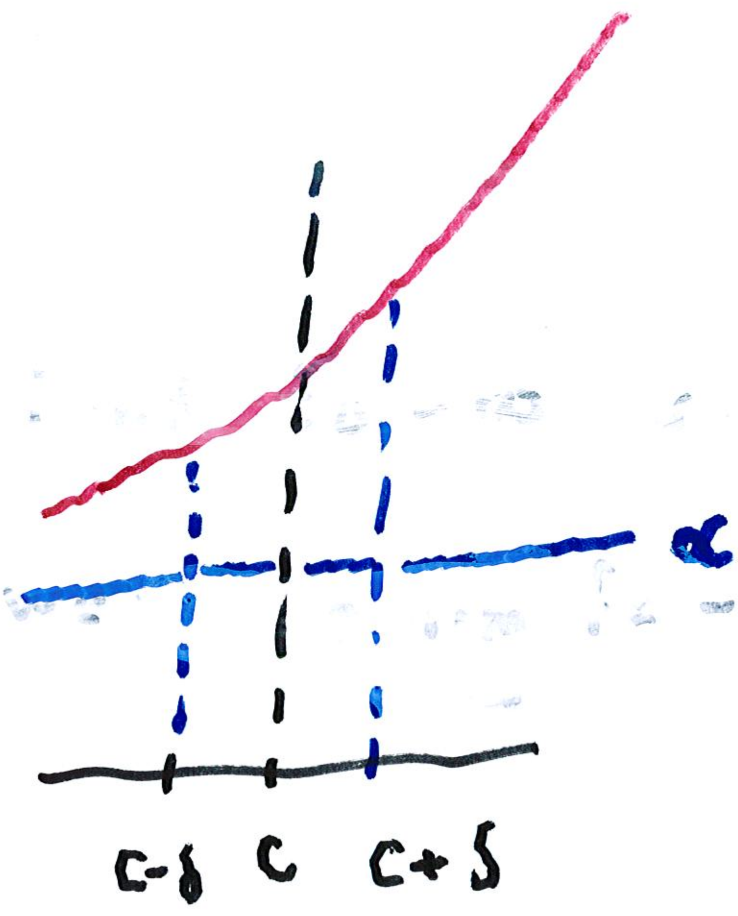
$f: A \to \mathbb{R}$, and let c be a

cluster point of A.

(i) We say f tends to ∞

as $x \to c$, and we write $\lim_{x \to c} f = \infty$

if for every $\alpha \in \mathbb{R}$ there is a $\delta = \delta(\alpha)$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.



Ex. Show $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$

Given $\alpha > 0$,

We need x to satisfy

$$\frac{1}{x^2} > \alpha \iff \frac{1}{\alpha} > x^2$$

or $\frac{1}{\sqrt{\alpha}} > |x|$

Working back, set $\delta = \frac{1}{\sqrt{\alpha}}$

I $0 < |x - 0| < \delta$, then

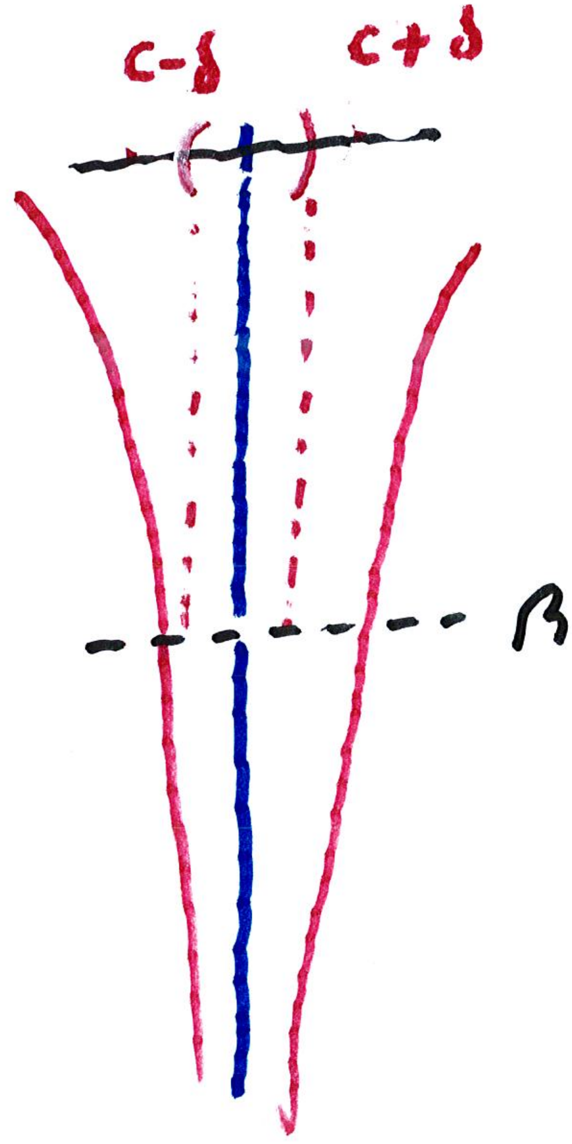
$$|x|^2 < \delta = \frac{1}{\sqrt{\alpha}}$$

$$\to \sqrt{\alpha} < |x|^2$$

(iii) We say f tends to $-\infty$

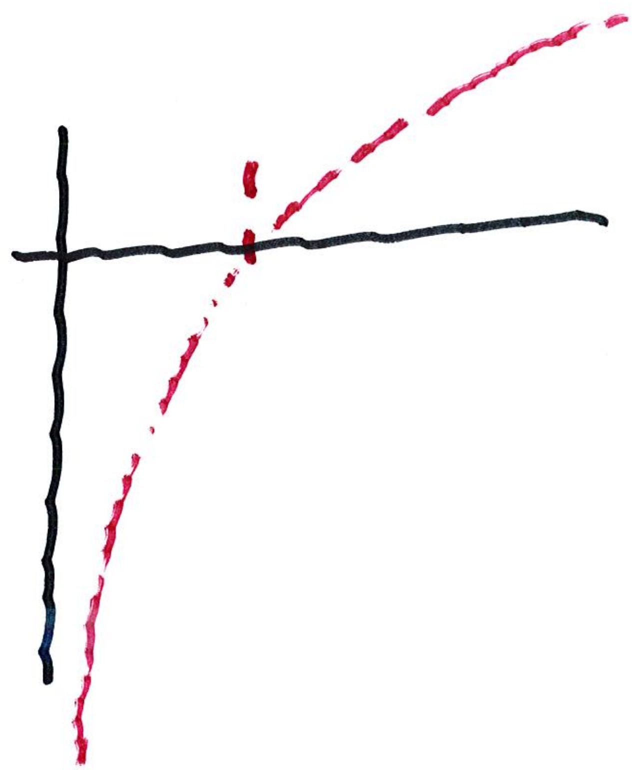
and we write $\lim\limits_{x \to c} f(x) = -\infty$

if for every $\beta \in \mathbb{R}$, there is a $\delta = \delta(\beta) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$



Ex. When we define $\ln x$,

we'll see $\lim\limits_{x \to 0} \ln x = -\infty$



We can also define limits $\to \infty$ f(x) as $x \to \infty$.

We say $\lim\limits_{x \to \infty} f = \infty$ if given any $\alpha > 0$, there is $K = K(\alpha)$ so that for any $x > K$, then

$f(x) > \alpha$.

$$\lim_{x \to \infty} \sqrt{x} = \infty$$

$$\lim_{x \to \infty} \ln x = \infty$$

5.1 Continuous Functions

Def'n Let $A \subseteq \mathbb{R}$, let

$f: A \to \mathbb{R}$ and let $c \in A$.

We say f is continuous at c

if, given any number $\varepsilon > 0$,

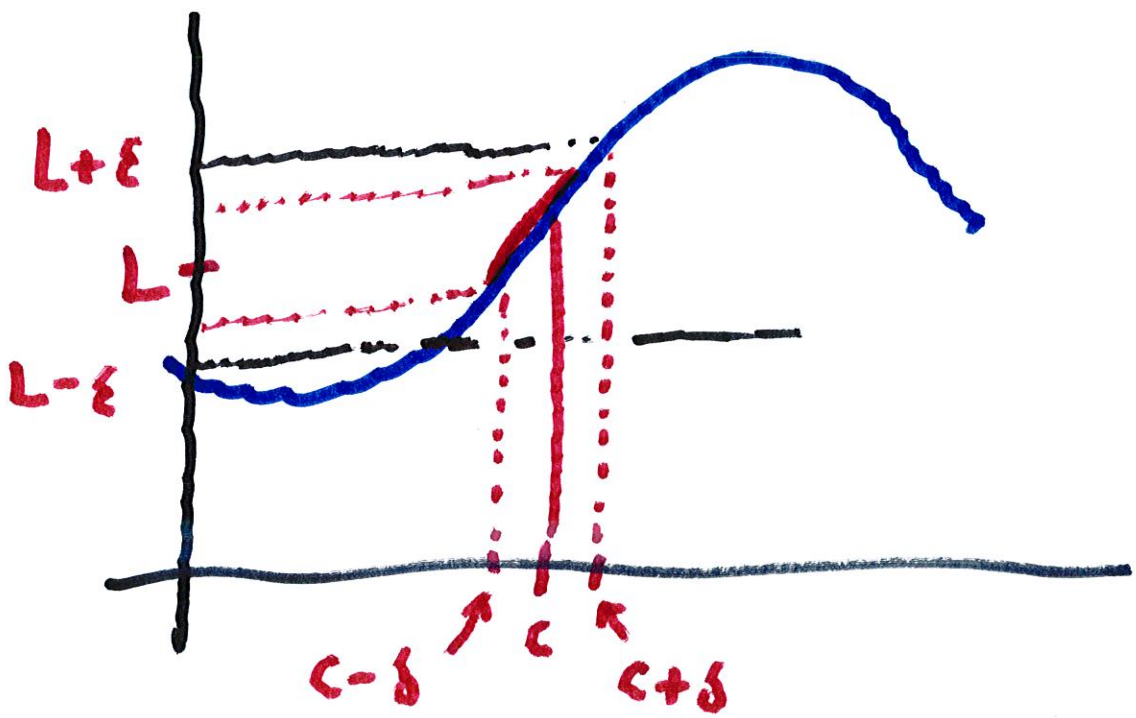
there exists $\delta > 0$ such that

if x is any point of A satisfying

$|x-c| < \delta$, then $|f(x) - L| < \varepsilon$

If f is not continuous at c,

then f is discontinuous at c.



Sequential Criterion for Continuity

A function $f: A \longrightarrow \mathbb{R}$ is

continuous at the point c in A

if and only if

for every every sequence (x_n)

in A such that (x_n) converges

to c, the sequence $(f(x_n))$

converges to $f(c)$

Discontinuity Criterion.

Let $A \subseteq \mathbb{R}$ and let $c \in A$.

Then f is discontinuous at c

if and only if

there exists a sequence

(x_n) in A such that (x_n)

converges to c, but the

sequence does not converge

to $f(c)$.

Def'n. Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$.

If B is a subset of A, we say that f is continuous at every point of B.

Examples

1. $g(x) = x$ is cont on $\mathbb{R}$

2. $h(x^2) = x^2$ is continuous on $\mathbb{R}$

3. We've shown $f(x) = \frac{1}{x}$

is continuous at $\{x \in \mathbb{R} ; x \neq 0\}$