Recall that a function

$f: A \to \mathbb{R}$ is continuous

at c if $\lim_{x \to c} f(x) = f(c)$.

If we define

$$V_\delta(c) = \left\{ x \in \mathbb{R} : |x - c| < \delta \right\}$$

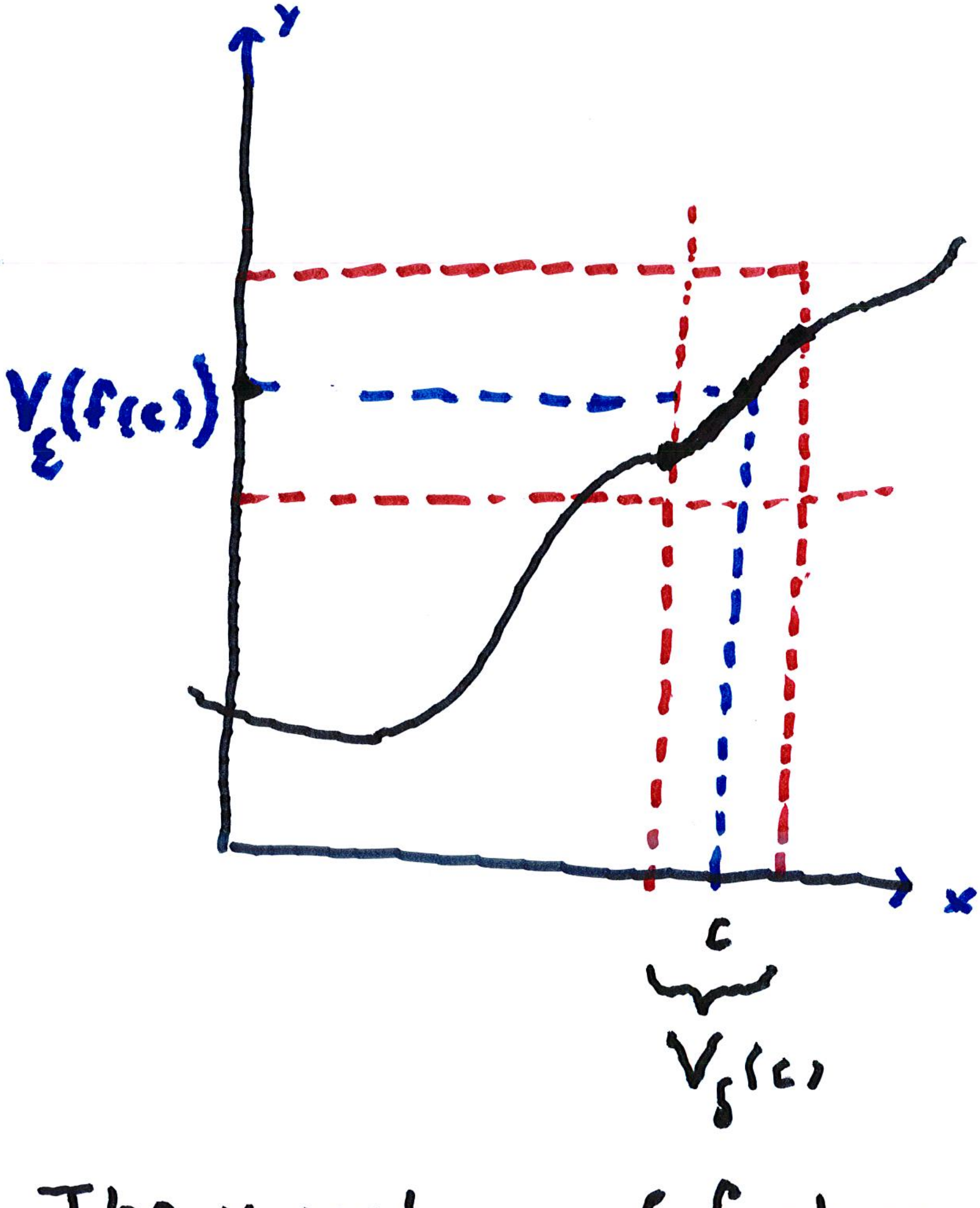
then we can write the

limit as follows:

A function $f: A \to \mathbb{R}$ is continuous at c if:

For every ε-neighborhood $V_\varepsilon(f(c))$, there is a δ-neighborhood $V_\delta(c)$ of c such that if x is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(f(c))$.

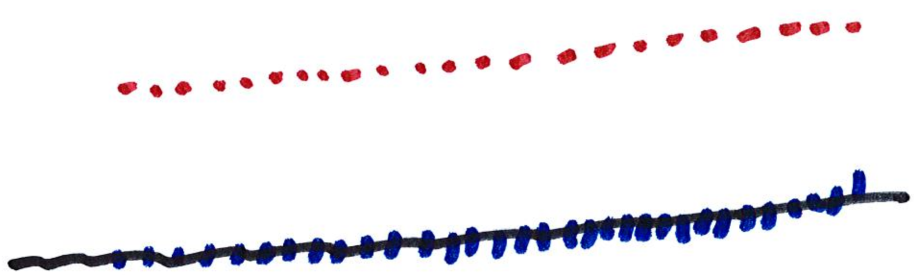
$\therefore \; f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$.



The y-values of f above $V_\delta(c)$ lie in $V_\varepsilon(f(c))$.

Some examples.

Set $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$



f is discontinuous at all x.

If c is irrational, then since $\mathbb{Q}$ is dense, for every $n > 0$ we can find a rational number x_n with $|x_n - c| < \frac{1}{n}$ and $x_n \neq c$

If f were continuous at c,

then $\lim f(x_n) = f(c)$,

i.e. $\lim 1 = 0$, contradiction.

Similarly suppose c is rational.

Since the irrationals

are dense, for every $n > 0$

we can find an irrational d_n

with $0 < |d_n - c| < \frac{1}{n}$.

If f were continuous at c.

then $\lim_{n\to\infty} f(d_n) = f(c)$, i.e

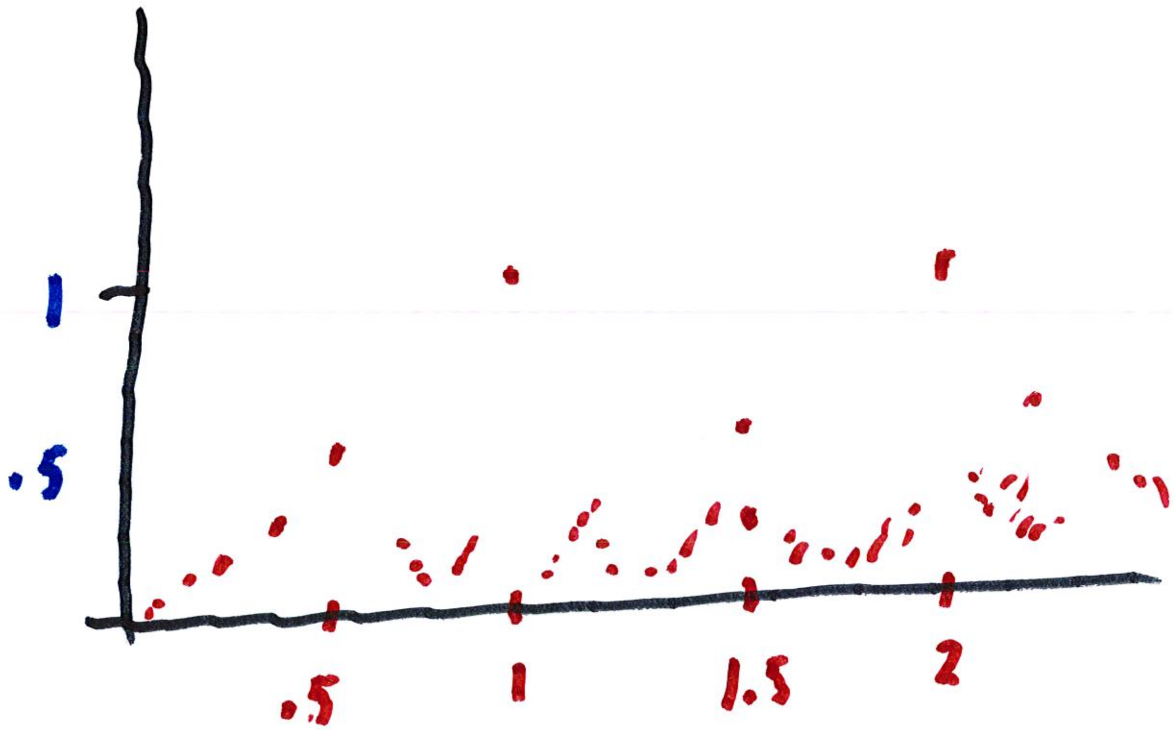
$\lim_{n\to\infty} (0) = 1$. Contradiction

Thomae's Fcn.

$$\text{Define } f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \pm \frac{p}{q} \end{cases}$$

where p, q have no common factor

Also, $q > 0$.



There is a good picture on pg. 127

First we show that f is discontinous at all rational numbers.

As in the previous example, let $c = \pm\frac{p}{q}$, so that $f(c) = \frac{1}{q}$.

The irrational numbers are dense so we choose a sequence of irrational numbers

x_n such that

$$0 < |x_n - c| < \frac{1}{n}.$$

If f were continous at c,

then $\lim f(x_n) = f(c) = \frac{1}{q}$

or $\lim 0 = \frac{1}{q}$.

$\therefore$ f is discontinuous at every rational number.

r_n in $(0,1)$ such that

$r_n = \frac{p}{q}$ and $q \le N$.

Choose $\delta > 0$ so that

all of the numbers r_n

lie outside the interval

$I = (c-\delta, c+\delta)$. Thus the

only rational numbers r

in I all have denominators

$\ge \frac{1}{N}$

Now we show that f is continuous at every irrational number c. We can assume that $0 < c < 1$. Let $\varepsilon > 0$, and choose N to be an integer with $\frac{1}{N} < \varepsilon$. There are only a finite number of rational numbers with $q > N$. It follows

that for $|x-c| < \delta$, we have

$$|f(x) - f(c)|$$

$$= |f(x)| = f(x) \le \frac{1}{N} < \varepsilon.$$

It follows that if

$$0 < |x-c| < \delta,$$

then $|f(x) - f(c)| < \varepsilon$.

Thus f is continuous at each irrational c.

5.2 Combinations of Continuous Functions

Recall that if

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M.$$

then

1. $\lim (f+g) = L + M$

2. $\lim (f-g) = L - M$

3. $\lim (fg) = LM$ 4. $\lim bf = bL$

5. if $M \neq 0$, then $\lim f/g = \dfrac{L}{M}$

When f and g are continuous at c, then

$$\lim_{x \to c} f(x) = f(c)$$ and

$$\lim_{x \to c} g(x) = g(c).$$ Hence,

1. $$\lim_{x \to c} (f+g) = f(c) + g(c)$$

2. $$\lim_{x \to c} (f-g) = f(c) - g(c)$$

3. $$\lim_{x \to c} (fg) = f(c)g(c)$$

4. $\lim_{x \to c} (bf) = b f(c)$

5. If $g(c) \neq 0$, then

$$\lim_{x \to c} \left(f/g \right) = f(c)/g(c)$$

This implies that $f+g$, $f-g$, fg, bf and f/g are all continuous at c.

(provided that $g(c) \neq 0$ in 5.)

It follows that any polynomial and also every rational function $R(x) = P(x)/Q(x)$ are continuous at every c (except when $Q(x) = 0$

We say a function f defined on A is continuous ~~at each~~ on A if f is continuous at each $c \in A$.

Composition of Continuous Fcns.

Suppose $f: A \to \mathbb{R}$ is continuous at c

and that $g: B \to \mathbb{R}$ is

continuous at $b = g(c)$.

then we'll show

$(g \circ f)(x) = g(f(x))$ is also

continuous at c, provided

$$f(A) \subseteq B.$$

More precicely:

Thm. Let $A, B \subseteq \mathbb{R}$ and

let $f: A \to \mathbb{R}$ and

$g: B \to \mathbb{R}$

be functions such that

$f(A) \subseteq B$. If f is continuous

at a point $c \in A$ and g is

continuous at $b = f(c) \in B$, then

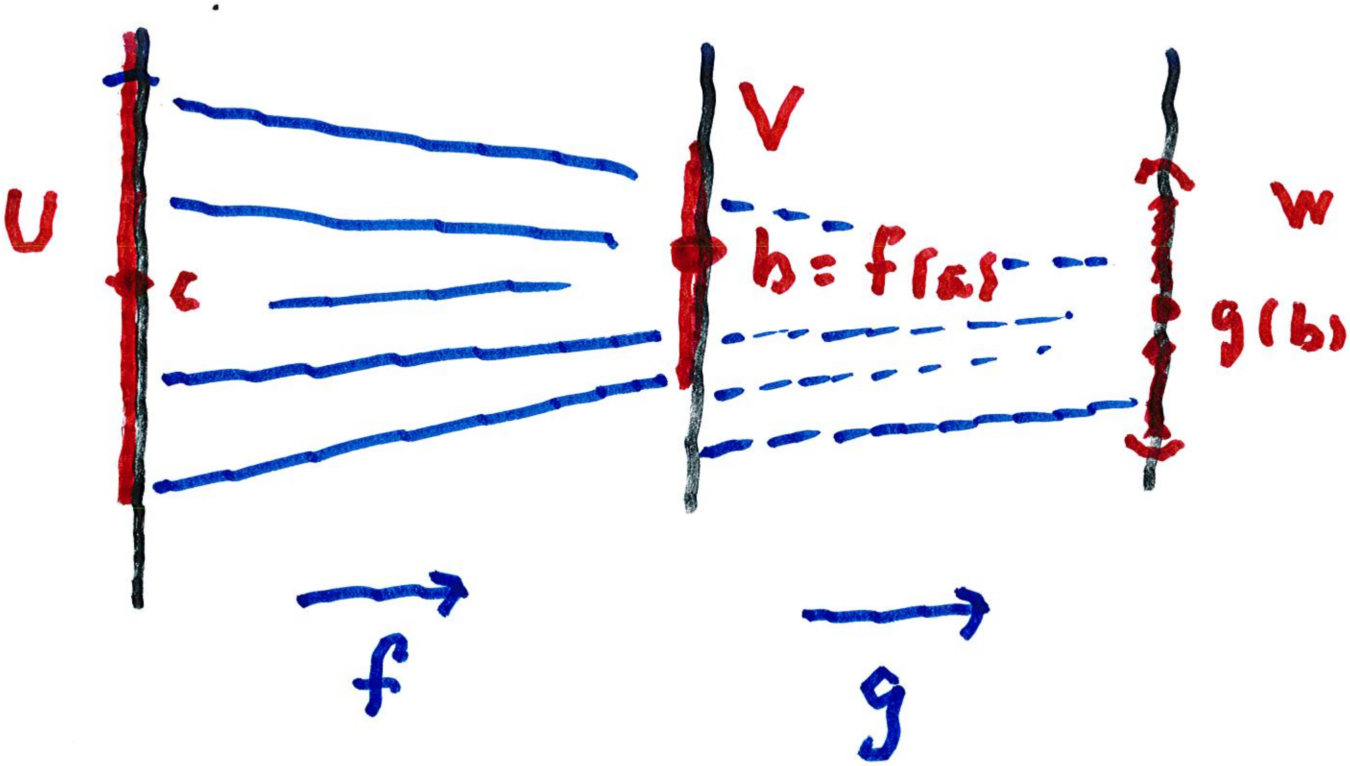
the composition $g \circ f$ is continuous at c.

Proof: Let W be an ε-neighborhood of $g(b)$. Since g is continuous at b, there is a δ-neighborhood V of $b = f(c)$ such that if $y \in B \cap V$ then $g(y) \in W$. Since f is continuous at c, there is a $\mathcal{Y}$-neighborhood U of c such that if $x \in A \cap V$, then $f(x) \in V$. Since $f(A) \subseteq B$, it follows that if $x \in A \cap U$, then $f(x) \in B \cap V$ so that $(g \circ f)(x) = g(f(x)) \in W$. But since W is an arbitrary

ε-neighborhood of $g(b)$, this

implies that $g \circ f$ is continuous

at c.



Thm

Let $f: A \to \mathbb{R}$ and let $g: B \to \mathbb{R}$ be continuous on A and B respectively. If $f(A) \subseteq B$, then $g \circ f : A \to \mathbb{R}$ is continuous on A.