Recall that we proved

Thm. Suppose that $f: A \to B$ and $g: B \to \mathbb{R}$, and that f is continuous at each $c \in A$, and that g is continuous at each point $b = f(c)$. Then $g \circ f$ is continuous at each point c in A.

Example: Suppose that $f: A \to \mathbb{R}$ is continuous on A. Then the function $|f(x)|$, $x \in A$ is continuous at each point $c \in A$.

First we show that the function $x \to |x|$ is continuous. We use 2.2.4 (b).

Let $x, c \in \mathbb{R}$.

$$|x| = |(x-c)+c| \leq |x-c| + |c|$$

$$\rightarrow |x| - |c| \leq |x-c|.$$

Also, $|c| = |(c-x)+x| \leq |x-c| + |x|$

$$\rightarrow |c| - |x| \leq |x-c|$$

Case 1. Suppose $|x| \geq |c|$.

Then $\big| |x| - |c| \big| \leq |x-c|$

Case 2. Suppose $|c| > |x|$

Then

$$\big|\,|x| - |c|\,\big| \leq -(|x| - |c|)$$

$$= |c| - |x| \leq |x - c|$$

$$\therefore \big|\,|x| - |c|\,\big| \leq |x - c|.$$

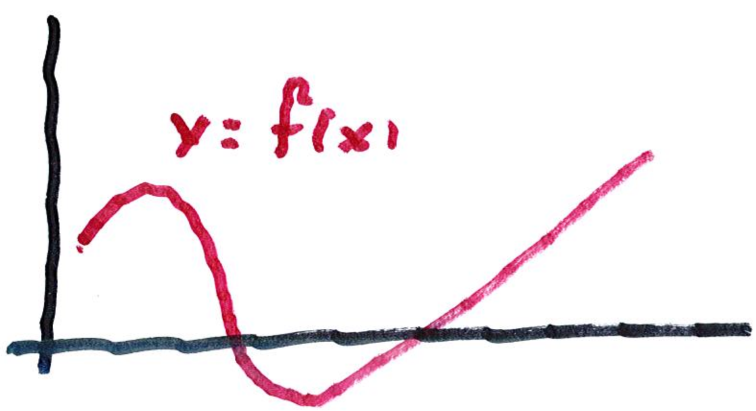
$\therefore$ the function $x \to |x|$ is

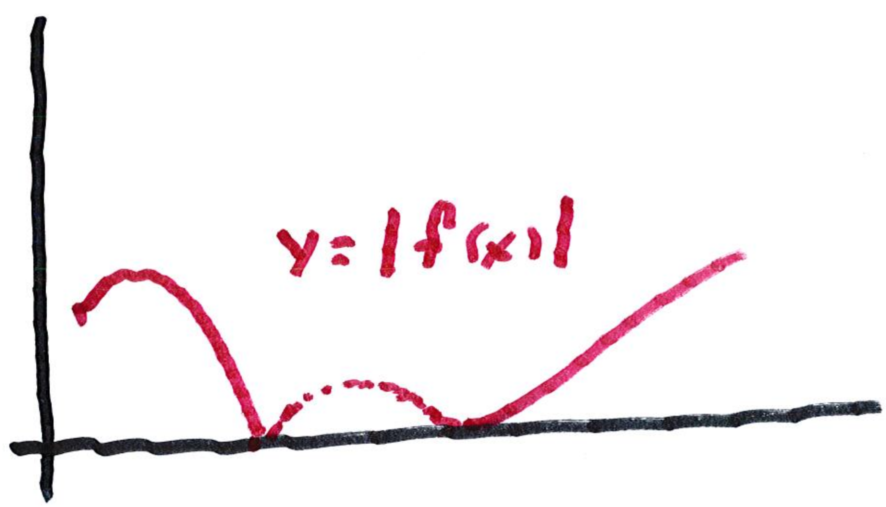
continuous at every $c \in \mathbb{R}$

By the Composition Thm.,

The function $|f(x)|$ is

continuous at every point $c \in A$.





5.3 Continuous Functions.

The set of continuous functions on a closed bounded interval has many special properties:

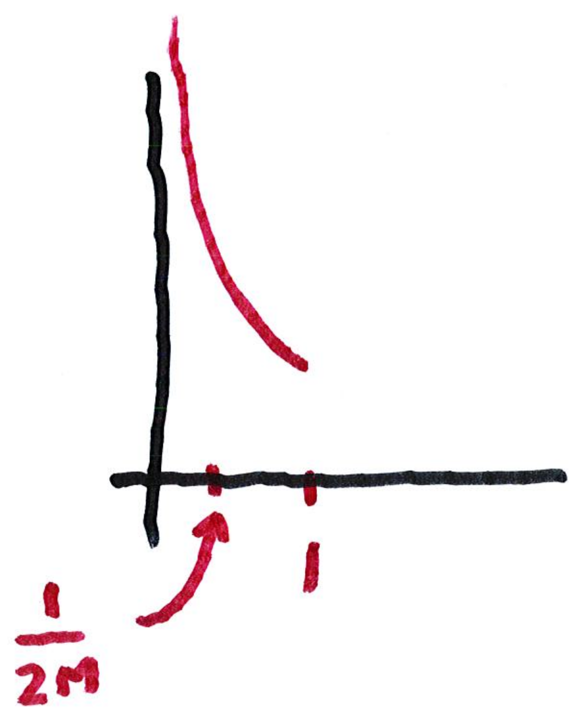
Def'n. A function $f: A \to \mathbb{R}$ is bounded on A if there is a constant $M > 0$ such that $|f(x)| \leq M$ for each $x \in A$.

The function $\frac{1}{x}$ is not bounded on $(0, 1]$. In fact, if we assume $\frac{1}{x} \leq M$. Then,

if we set $x_M = \frac{1}{2M}$,

then $\frac{1}{\frac{1}{2M}} = 2M > M$.

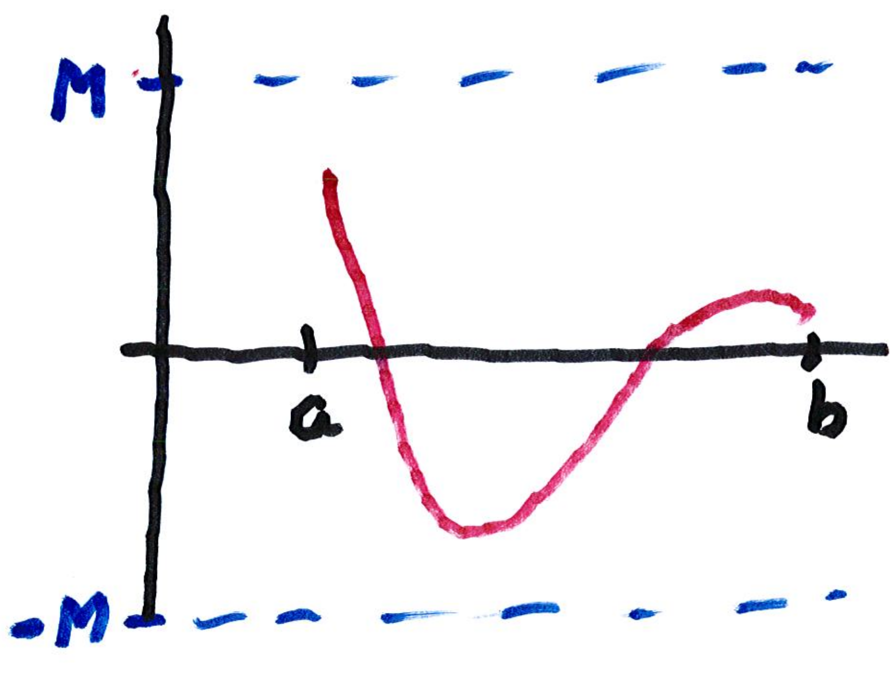
which is a contradiction



Boundedness Thm.

Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I.

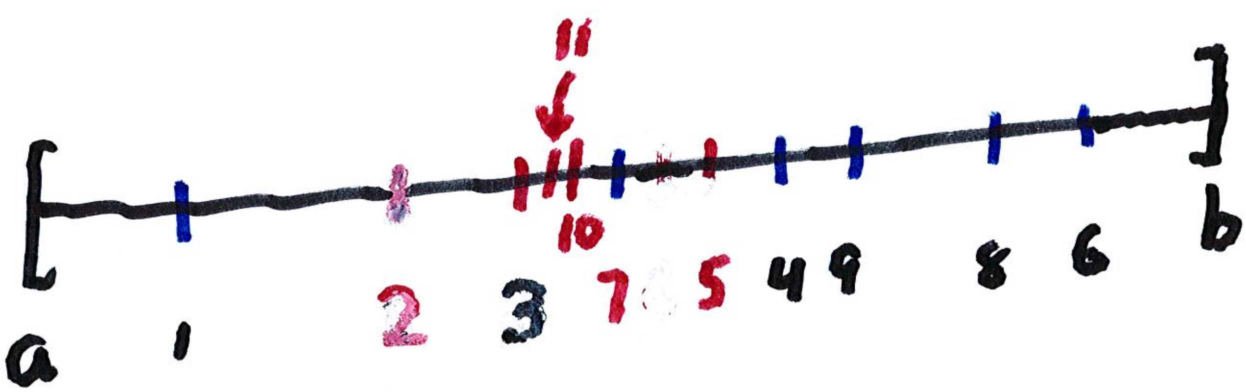
Then f is bounded on I



Proof. Suppose that f is NOT bounded on I. Then for any $n \in \mathbb{N}$, there is a number $x_n \in I$ such that $|f(x_n)| > n$.

Since I is bounded, the sequence $X = (x_n)$ is bounded.

Therefore the Bolzano-Weierstrass Theorem implies there is a subsequence $X' = (x_{n_k})$ of X that converges to x.



Subsequence is

$x_2, x_5, x_7, x_{10}, x_{11}, \ldots$

Since $x_{n_k} \in [a, b]$ and

(x_{n_k}) converges to x,

it follows that $a \le x \le b$,

so $x \in I$. Then f is

continuous at x, which implies

that $(f(x_{n_k}))$ converges to $f(x)$.

But f is continuous at

x, so $|f|$ is bounded by M

in a small δ-neighborhood

of x. This leads to a contradiction, since

$$|f(x_{n_n})| > n_n \geq n, \qquad \text{all } n \in \mathbb{N}.$$

We only have to choose n so $|x_{n_n} - x| < \delta$ and so

$$n > M.$$

Def'n. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. We say f has an absolute maximum on A if there is a point $x^* \in A$ such that $f(x) \le f(x^*)$, for all $x \in A$

Similarly f has an absolute minimum on A if there is a point $x_* \in A$ such that $f(x) \ge f(x_*)$ for all $x \in A$.

On the interval $I = [a, b)$,

$f(x) = x$ does not have an absolute maximum on I.

Clearly $f(x) < b$ on I, but there is no point $\bar{x}$ in I such that $f(\bar{x}) = b$.

Thm. If f is continuous on $[a, b]$, then f has an absolute maximum at x^* and an absolute minimum at x_*.

Proof. We first show that there is an absolute maximum at a point x^*.

$$\text{Let } S = \left\{ f(x) : x \in [a, b] \right\}.$$

By the Boundedness Theorem there is a number $M > 0$ that is an upper bound of S.

By the Least Upper Bound Property, there is a least upper bound α of S.

Then $f(x) \leq \alpha$ for all $x \in [a, b]$. If there is an $x \in [a, b]$, such that $f(x) = \alpha$, then we are done. Just set $x^* = x$.

Suppose there is no $x \in [a, b]$ with $f(x) = \alpha$, then

$f(x) < \alpha$, all $x \in [a, b]$.

Set $g(x) = \dfrac{1}{\alpha - f(x)}$.

Since $f(x) < \alpha$, it follows that g is continuous on $[a, b]$.

On the other hand,

since $\alpha = \sup. S$,

for every $n \in N$, there

is an x_1 so that

$$\alpha - \frac{1}{n} < f(x_1) < \alpha$$

$$\alpha - f(x_1) < \frac{1}{n}$$

$$g(x_1) = \frac{1}{\alpha - f(x_1)} > n$$

Thus g is continuous on

$[a, b]$ and unbounded, which

contradicts the Boundedness Theorem. Hence there must be an $x = x^*$ such that $f(x^*) = \alpha$.

For the Absolute Minimum, set $h(x) = -f(x)$. We've just shown there is a point $x_* \in [a,b]$, so that $h(x_*) \geq h(x)$, all $x \in I$.

or $-f(x_*) \geq -f(x)$

or $f(x_*) \leq f(x)$, $x \in I$.