We've proved:

Boundedness Thm.

If f is continuous on $[a,b]$ (i.e., f is continuous at each $c \in [a,b]$), then there is $M > 0$ such that

$|f(x)| \leq M$, all $x \in [a,b]$

Also, we proved the

Maximum - Minimum Thm,

If f is continuous at each point c in $[a,b]$, then there are two points x^* and x_* in $[a,b]$, such that

$$f(x_*) \le f(x) \le f(x^*),$$

for all $x \in [a,b]$.

Now we prove the

Location of Roots Theorem.

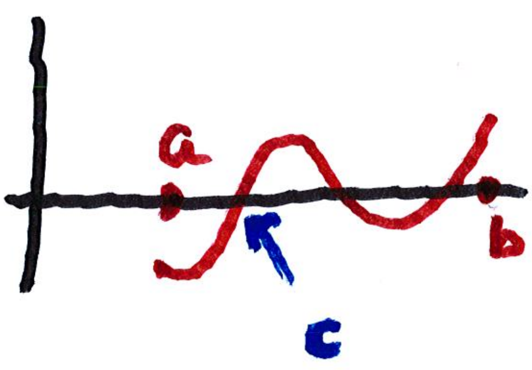
Let $I = [a,b]$ and let

$f : I \longrightarrow \mathbb{R}$ be continuous

on I. If $f(a) < 0 < f(b)$,

Then there is a point $c \in (a,b)$

such that $f(c) = 0$.



The proof is similar if $f(a) > 0 > f(b)$.

Pf. We use the Bisection Method. Set $a_1 = a$ and $b_1 = b$, and set $p_1 = \dfrac{a_1 + b_1}{2}$.

There are 3 cases:

Case 1. If $f(p_1) = 0$. Then set $c = p_1$. Then we are done.

Case 2. If $f(p_1) > 0$.

Then set $a_2 = a_1$ and $b_2 = p_1$.

Case 3. If $f(p_1) < 0$. Then set

$a_2 = p_1$ and $b_2 = b_1$.

In both Case 2 and Case 3,

we have $f(a_2) < 0 < f(b_2)$.

Assume by induction that we

have constructed $a_1, a_2, \ldots, a_k$

and $b_1, b_2, \ldots, b_k$

such that $f(a_k) < 0 < f(b_k)$.

Then set $I_k = [a_k, b_k]$.

Then set $p_k = \dfrac{a_k + b_k}{2}$.

As above, there are 3 cases.

Case 1. If $f(p_k) = 0$, then set $c = p_k$. Since $f(c) = 0$, we are done.

Case 2. Suppose $f(p_k) > 0$.

Then set $a_{k+1} = a_k$ and

set $b_{k+1} = p_k$

Case 3. If $f(p_k) < 0$. Set $a_{k+1} = p_k$ and $b_{k+1} = b_k$.

In both Case 2 and Case 3, we have $f(a_{k+1}) < 0 < f(b_{k+1})$, which completes the induction step.

If the process never terminates, then for every $n \in \mathbb{N}$, we have $f(a_n) < 0 < f(b_n)$.

Also, set $I_n = [a_n, b_n]$.

Clearly, the length of

$I_n = (b_n - a_n)/2^{n-1}$ for all

$n \in N$.

Since $I_1 \supset I_2 \supset \dots I_n \supset \dots$,

the Nested Interval

Property implies there is

a point c that belongs

to each interval I_n,

for $n \in N$.

Since $a_n \le c \le b_n$,

we have

$$0 \le c - a_n \le b_n - a_n = (b-a)/2^{n-1}$$

and

$$0 \le b_n - c \le b_n - a_n = (b-a)/2^{n-1}$$

The Sqeeze Thm. implies that

$$\lim a_n = c = \lim b_n$$

Since f is continuous at c,

we have

$\lim(f(a_n)) = f(c) = \lim(f(b_n))$.

Since $f(a_n) < 0$ for all $n \in \mathbb{N}$

, we have

$f(c) = \lim(f(a_n)) \leq 0$

Similarly, since $f(b_n) > 0$

for all n, we have

$f(c) = \lim(f(b_n)) \geq 0$.

Hence $f(c) = 0$.

In the case when

$f(a) > 0 > f(b)$,

we set $h(x) = -f(x)$, then

$h(a) = -f(a) < 0$ and

$h(b) = -f(b) > 0$.

Then the above theorem implies that there is a point $c \in (a,b)$ so that

$h(c) = 0 \rightarrow f(c) = 0$

We can generalize the theorem.

Bolzano's Intermediate Value Thm.

Let f be an interval and let $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$, and if $f(a) < k < f(b)$, then there is a point c between a and b such that $f(c) = k$.

Pf. Case 1. Suppose $a < b$

and let $g(x) = f(x) - k$.

Then $g(a) < 0 < g(b)$.

The Location of Roots Thm

implies there is c with

$a < c < b$ such that

$$0 = g(c) = f(c) - k.$$

Case 2. Suppose $b < a$, and

let $h(x) = k - f(x)$ so that

$h(b) < 0 < h(a)$. Hence

there is a point c so that

$0 = h(c) = k - f(c)$,

which implies $f(c) = k$.

Thm. Let I be a closed

interval and let $f: I \to \mathbb{R}$

be continuous on I.

Then $f(I) = \{ f(x) ; \; x \in I \}$

is a closed bounded

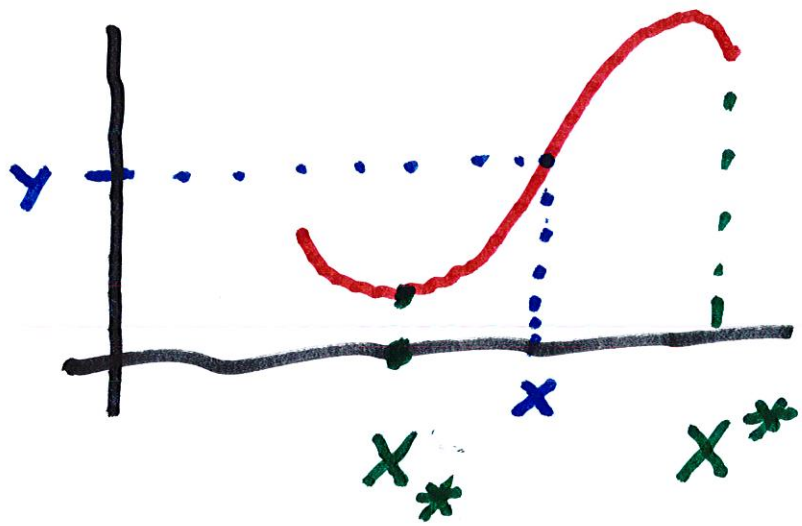
interval.

Pf. If we let $I = [a, b]$,

then there exist x_* and

x^* such that there is x with

$$f(x_*) \le f(x) \le f(x^*)$$



The Int. Value Thm. shows that for every y between $f(x_*)$ and $f(x^*)$, y is in $f(I)$. there is x between x_* and x^* so $f(x) = y$.

5.4. Uniform Continuity

Def'n. Let $f: A \longrightarrow \mathbb{R}$.

We say that f is

uniformly continuous on A

if for any $\varepsilon > 0$, there is

a $\delta(\varepsilon) > 0$ so that if

$x, u \in A$ satisfy $|x-u| < \delta$,

then $|f(x) - f(u)| < \varepsilon$.

The main point is that

$\delta = \delta(\varepsilon)$ only depends on ε.

Ex. $f(x) = \frac{1}{x}$, $0 < x \le 1$

Let $x = \varepsilon$ and $u = \frac{\varepsilon}{2}$

Then $|x - u| = \left|\varepsilon - \frac{\varepsilon}{2}\right| = \frac{\varepsilon}{2}$.

But $\left|\frac{1}{\varepsilon} - \frac{2}{\varepsilon}\right| = \frac{1}{\varepsilon}$. Even

though x and u are close,

$f(x)$ and $f(u)$ are not close.

Uniform Continuity Thm.

Let I be a closed bounded interval and let $f: A \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Proof. Suppose f is NOT uniformly continuous on I. Then for some $\varepsilon_0 > 0$, there are sequences (x_n) and (u_n)

in A such that

$$|x_n - u_n| < \frac{1}{n} \quad \text{and}$$

$$|f(x_n) - f(u_n)| \geq \varepsilon_0.$$

Since I is bounded, the sequence (x_n) is bounded. The Bolzano-Weierstrass Thm. implies there is a subsequence (x_{n_k})

of (x_n) that converges to an element z. (1)

Since $(x_n) \in I$ for all n, we see that (x_{n_k}) lies in $I = [a, b]$ for all k. Hence $z = \lim_{k\to\infty} (x_{n_k})$ also lies in I. Note that the sequence (v_{n_k}) satisfies

$$|u_{n_k} - z| \le |u_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

The first term on the right $\to 0$

since $|u_n - x_n| < \frac{1}{n}$.

The second term $\to 0$ by (1).

It follows that $\lim\limits_{k\to\infty} u_{n_k} = z$.

Since f is continuous at z,

we obtain $\begin{cases} f(x_{n_k}) \to f(z) \\ f(u_{n_k}) \to f(z) \end{cases}$

Hence $\lim |f(x_{n_k}) - f(u_{n_k})| = 0$

But this is not possible since $|f(x_n) - f(u_n)| \geq \varepsilon_0$.

This contradiction shows that f is uniformly continuous on I.