Uniform Continuity (cont'd)

A function $f: A \to \mathbb{R}$ is

<u>uniformly continuous</u> if

for all $\varepsilon > 0$, there is a

$\delta = \delta(\varepsilon)$ such that if

x and u are in A and

satisfy $\underline{|x - u| < \delta}$, then

$\underline{|f(x) - f(u)| < \varepsilon}$.

Thm. Let $I = [a, b]$, and suppose that f is continuous on I. Then f is uniformly continuous on I.

Proof (by contradiction)

Suppose f is NOT uniformly continuous. Then there is a fixed $\varepsilon_0 > 0$ and also two sequences (x_n) and (u_n) in I such that

$$|x_n - u_n| < \frac{1}{n} \text{ and}$$

$$|f(x_n) - f(u_n)| \geq \varepsilon_0. \quad (1)$$

Since I is bounded, the sequence (x_n) is bounded.

Hence the Bolzano-Weierstrass Theorem implies there is a subsequence $\{x_{n_k}\}$ of (x_n) that converges to an element z.

Since $x_{n_k} \in I$, we have $x_{n_k} \geq a$. Hence the limit z is also $\geq a$. Similarly $z \leq b$. Thus $z \in I$.

The sequence u_{n_k} also converges to z since

$$|u_{n_k} - z| \leq |u_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

In fact $|u_n - \ n| < \frac{1}{n} \to 0$

and x_{n_k} converges to z as $k \to \infty$.

Since f is continuous at z,

we have $\left\{ \begin{array}{l} f(x_{n_k}) \to f(z) \\ f(u_{n_k}) \to f(z). \end{array} \right.$

Hence $|f(x_{n_k}) - f(u_{n_k})|$

converges to 0, which

contradicts (1).

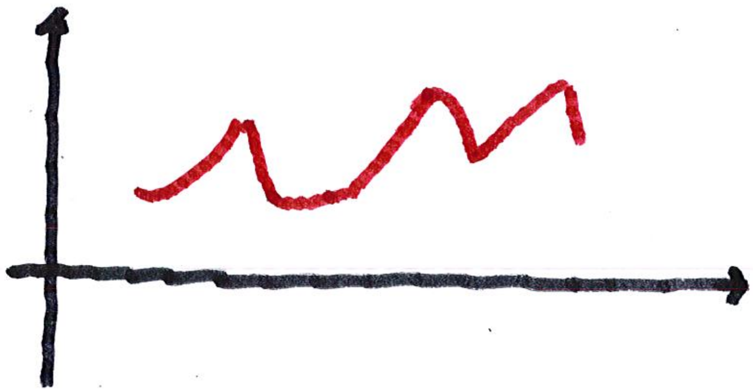
It follows that f <u>is</u>

uniformly continuous.

Def'n. Let $f: A \to \mathbb{R}$. Then

f is Lipshitz if for all x, u

in A, $|f(x) - f(u)| \leq K|x-u|$



The function $f(x) = \sqrt{x}$,

for $x \geq 0$

is NOT Lipshitz (for any K)

Set $u = 0$. If f is Lipshitz,

$$\left| \sqrt{x} - 0 \right| \leq K|x - 0|$$

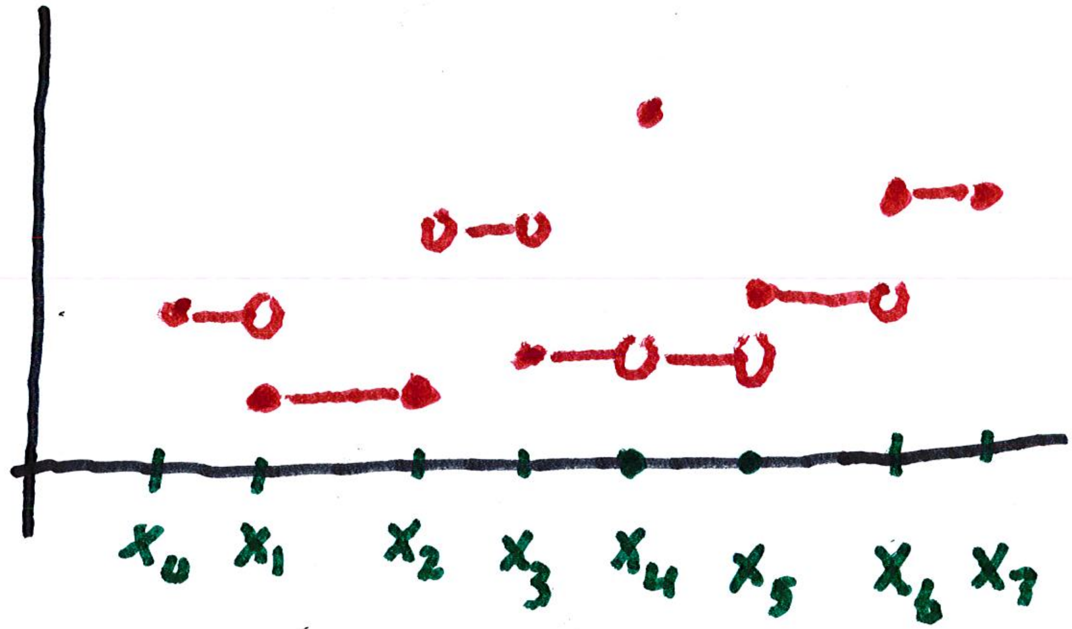
$$\rightarrow \quad \sqrt{x} \leq Kx \quad \rightarrow 1 \leq K\sqrt{x}$$

A Lipshitz function is uniformly continuous, because if $|x-u| < \frac{\varepsilon}{K}$,

then

$$|f(x)-f(u)| \le K|x-u| < \frac{K\varepsilon}{K} = \varepsilon.$$

Step Functions. Suppose that $\{a = x_0 < x_1, \ldots, x_{N-1} < x_N\}$



A step function

The width of each step can vary.

A step function is constant on each interval

(x_{k-1}, x_k), for $k = 1, 2, \dots, N$.

Approximation Theorem.
We can always approximate a continuous function on $[a, b]$ by a step function.

Given $\varepsilon > 0$, find $\delta > 0$ so that if $|x-u| \le \delta$, then

$|f(x) - f(u)| < \varepsilon$

Assume the partition

$\{x_0, x_1, \ldots, x_N\}$ satisfies

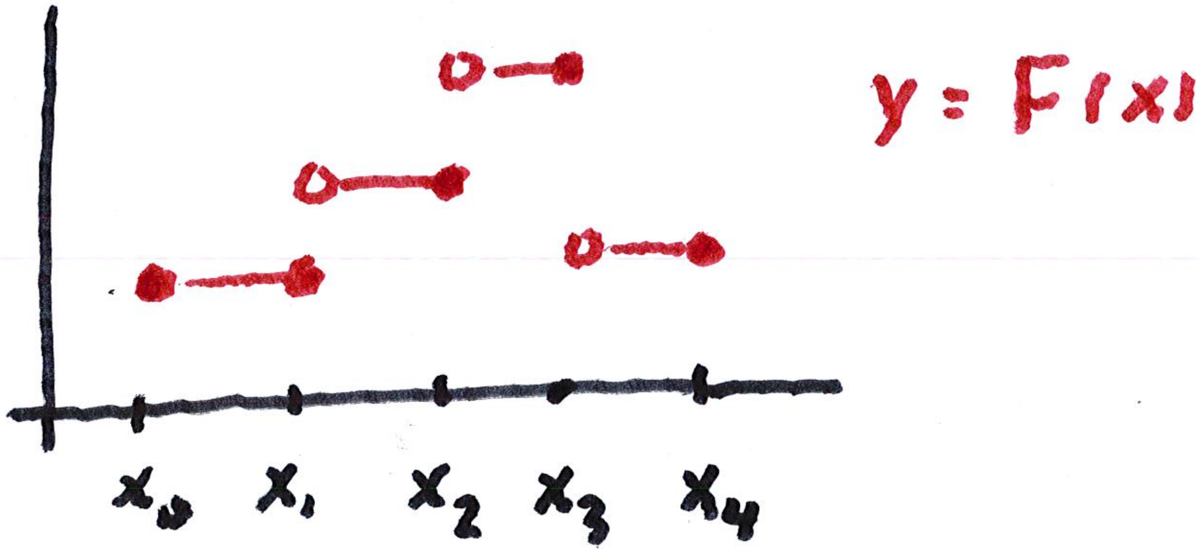
$|x_k - x_{k-1}| < \delta$ for all

$k = 1, 2, \ldots, N.$

Define $F(x) = f(x_k)$

if $x_{k-1} < x \leq x_k$.

and define $F(x_0) = f(x_1)$.



We show $|F(x) - f(x)| < \varepsilon$

if $x \in [a, b]$

When $x_{k-1} < x \leq x_k$,

$|F(x) - f(x)| = |f(x_k) - f(x)|$

$< \varepsilon$, since $|x_k - x| < \delta$.

We still need to consider

the case when $x = x_0$.

we have

$$|F(x_0) - f(x_0)| = |f(x_1) - f(x_0)| < \varepsilon$$

which shows that for all

$x \in [a, b]$,

$$|F(x) - f(x)| < \varepsilon.$$