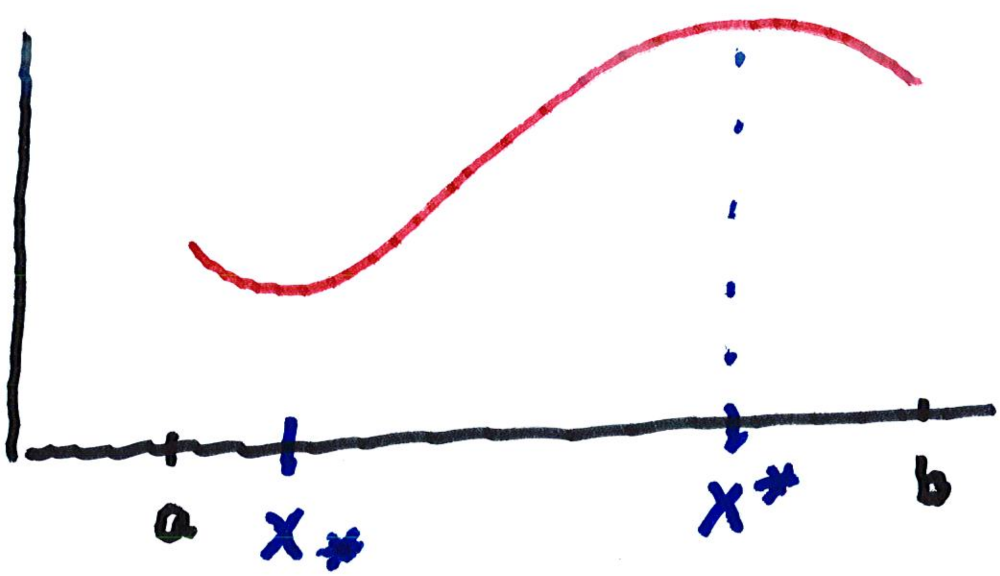
If f is continuous on a closed bounded interval $I = [a, b]$, we showed that there are 2 points x_* and x^* such that $f(x_*) \le f(x) \le f(x^*)$.



We will say that a function f is strictly increasing on an interval I if whenever $x' < x''$ then $f(x') < f(x'')$.

Let's assume that f is strictly increasing and continuous on $[x_*, x^*]$.

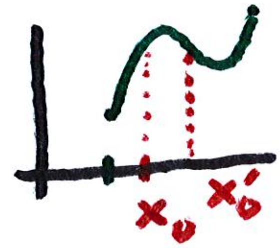
Suppose that k satisfies

$$f(x') < k < f(x'')$$

Then the Intermediate Value Thm (IVT) says that

there is a number $x_0 \in (x_*, x^*)$

such that $f(x_0) = k.$



In fact, this x_0 must be unique, for if x_0' is another number with $f(x_0) = k = f(x_0')$, then f would not be strictly increasing. Hence x_0 is unique.

We can define the inverse

by setting $g(y) = x$,

whenever $f(x) = y$.

Thus the function $g(y)$ is

well-defined for all y

that satisfy

$$f(x_*) \leq y \leq f(x^*)$$

Note that if $x \in [x_*, x^*]$,

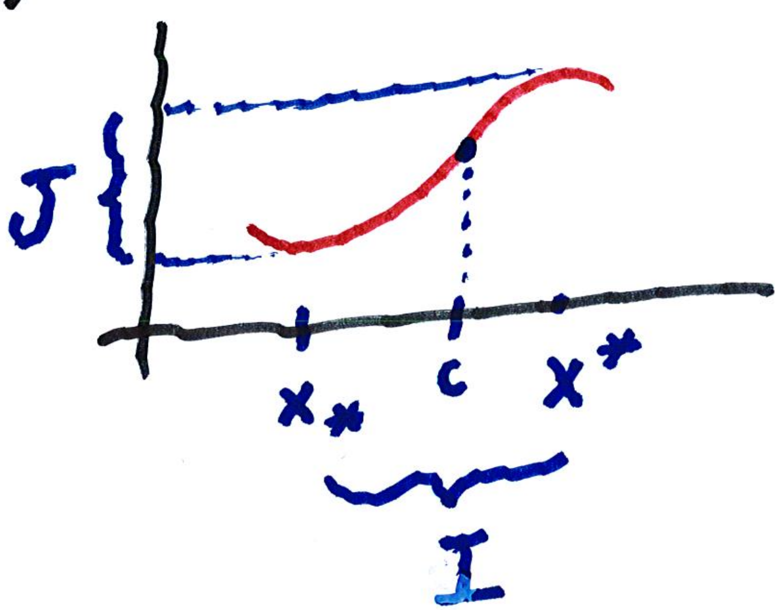
then $f(x) \in [f(x_*), f(x^*)]$

$= J$. If we set $y = f(x)$,

then $y \in$ Range of f.

Thus $g(y) = x$, so

$J = f(I)$



We want to show that the inverse function g is continuous on J.

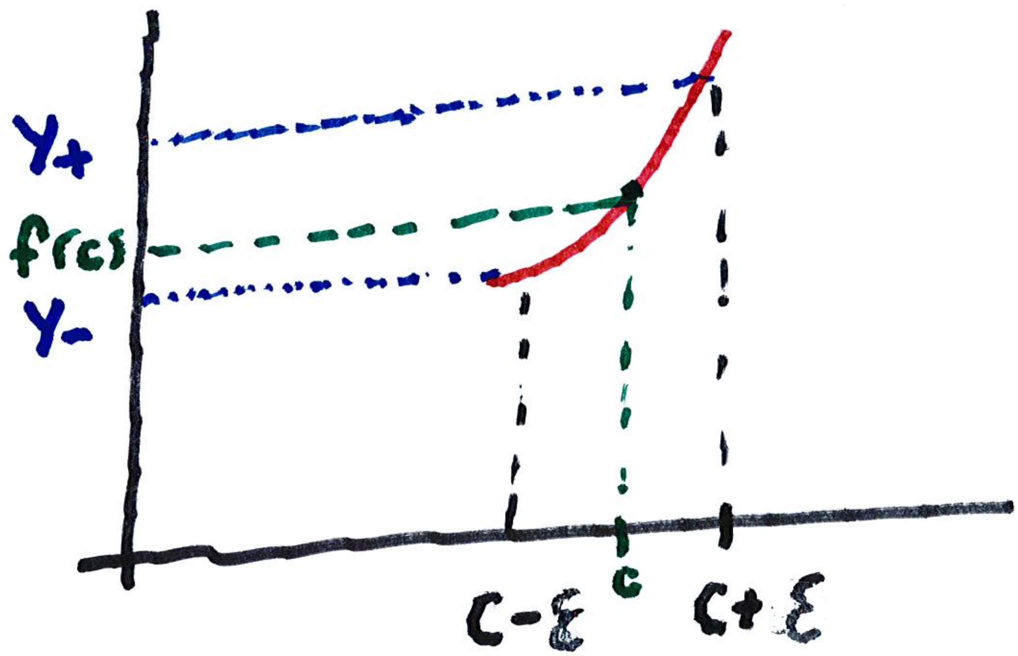
Let $c \in I$. For any small number ε, set $y_+ = f(c+\varepsilon)$ and set $y_- = f(c-\varepsilon)$.

This implies

$$g(y_+) = c+\varepsilon \text{ and } g(y_-) = c-\varepsilon$$

If $c - \varepsilon < x < c + \varepsilon$, then

$y_- < f(x) < y_+$.



Now set

$$\delta = \min\left\{ |y_+ - f(c)|,\ |y_- - f(c)| \right\}$$

It follows that if

$y \in V_\delta(f(c))$, then

$g(y) \in V_\varepsilon(c)$.

It follows that g is continuous

at $f(c)$. Since c is arbitrary,

it follows that

$g : [f(a), f(b)] \to [a, b]$ is

continuous at c.

Thus we've proved that if

f is continuous on an

interval I, and if f

is strictly increasing on I.

then there is a continuous

function g on $J = [f(a), f(b)]$

such that

$$g(f(x)) = x. \quad \text{all } x \in [a,b].$$

6.1 The derivative.

Def'n. Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$, and let $c \in I$. We say that L is the derivative of f at c if given any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that if $x \in I$ and satisfies

$0 < |x-c| < \delta(\varepsilon)$, then

$$\left| \frac{f(x)-f(c)}{x-c} - L \right| < \varepsilon.$$

We write $f'(c) = L$

Thus the derivative of f at c is given by

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

A useful theorem:

Thm. If $f: I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Pf. For all $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right)(x - c).$$

Since $\lim \left(\frac{f(x) - f(c)}{x - c} \right)$ and

$\lim (x - c)$ exist, the

product rule implies that

$$\lim_{x \to c} (f(x) - f(c))$$

$$= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) \cdot \lim_{x \to c} (x - c)$$

$$= f'(c) \cdot 0 = 0$$

$$\therefore \lim_{x \to c} f(x) = f(c).$$

This shows that f is continuous at c

These limit laws are very important:

Thm. Suppose that both f and g are differentiable at $c \in I$. Then:

(a) $(bf)'(c) = bf'(c)$

(b) $(f+g)'(c) = f'(c) + g'(c)$.

(c) Product Rule.

$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(d) Quotient Rule If $g(c) \neq 0$,

then $$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

We'll prove the Product and Quotient Rules:

(c) (Prod. Rule) Let $p = fg$.

$$\frac{p(x) - p(c)}{x - c} = \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x) - f(c)}{x-c}\, g(x) + f(c) \cdot \frac{g(x) - g(c)}{x-c}.$$

Since $g(x)$ is differentiable at c, it's also continuous at c.

$$\lim \frac{p(x) - p(c)}{x-c} = f'(c)g(c) + f(c)g'(c)$$

The Quotient Rule is:

$$\left(\text{Set } q(x) = \frac{f(x)}{g(x)}\right)$$

Since g is differentiable, it's also continuous at c. Hence $g(x) \neq 0$ in a neighborhood of c (since $g(c) \neq 0$)

$$\frac{q(x) - q(c)}{x - c} = \frac{f(x)/g(x) - f(c)/g(c)}{x - c}$$

$$= \frac{f(x)\,g(c) - f(c)\,g(x)}{g(x)\,g(c)\,(x - c)}$$

$$= \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{1}{g(x)g(c)} \left[\frac{f(x)-f(c)}{x-c}\, g(c) - f(c)\, \frac{g(x)-g(c)}{x-c} \right]$$

Using the continuity of g, we get

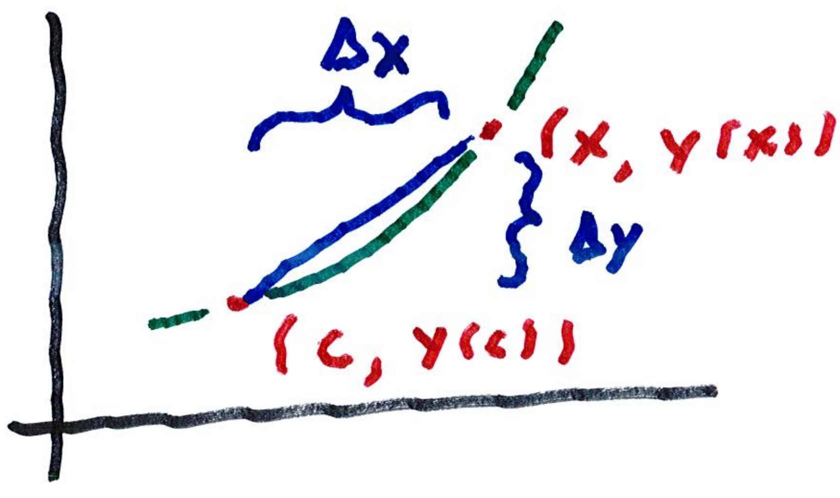
$$g'(c) = \lim \frac{g(x)-g(c)}{x-c}$$

$$= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

If we use the notation

$$\frac{d}{dx}\left(y(x) \right) = \lim_{x \to c} \frac{y(x) - y(c)}{x - c}$$

we get $\displaystyle = \lim_{x \to c} \frac{\Delta y}{\Delta x}$



As $x \to c$, $\Delta x = x - c \to 0$

$\therefore \lim \frac{\Delta y}{\Delta x}$ = slope of the tangent line at $(c, y(c))$

The Chain Rule

Carathéodory's Theorem.

Let f be defined on an interval I containing c. Then f is differentiable at c if and only if there is a function φ on I that is continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)(x - c), \quad \text{for } x \in I \qquad (1)$$

In this case $\varphi(c) = f'(c)$.

$\Rightarrow$ If $f'(c)$ exists, we can define φ by

$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{for } x \neq c,\ x \in I \\ f'(c) & \text{for } x = c \end{cases}$$

The continuity of φ follows from the fact that $\lim_{x \to c} \varphi(x) = f'(c)$

If $x = c$, then both sides of (1) equal 0, and if $x \neq c$, then multiplication of $\varphi(x)$ by $x-c$ gives (1).

$\Leftarrow$ Assume that a function φ that is continuous at c and satisfying (1) exists.

If we define (1) by $(x-c) \neq 0$, then the continuity of φ implies that

$$\varphi(c) = \lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$\therefore$ f is differentiable at c

and $f'(c) = \varphi(c)$.