6.2 The Mean Value Theorem

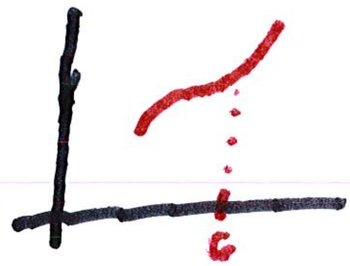
We say that a function f defined on an interval I has a relative maximum at $c \in I$ if there is a neighborhood of c such that $f(x) \leq f(c)$ for all x in $I \cap V_\delta$. Similarly f has

a relative minimum if $f(x) \geq f(c)$ in that neighborhood.

If f has either a relative maximum or minimum, then we say f has a relative extremum in such a neighborhood.

Interior Extremum. Let c be a relative extremum of a function f in the interior of an interval I. If the derivative of f at c exists,

then $f'(c) = 0$.



Proof. Suppose first that f has a relative maximum at c.

If $f'(c) > 0$, then set

$\varepsilon = \dfrac{f'(c)}{2}$. If δ is sufficiently

small, say $c < x_1 < c+\delta$,

we have

$$\frac{f(x_1) - f(c)}{x_1 - c} > \frac{f'(c)}{2}.$$

Hence $f(x_1) > f(c) + \dfrac{f'(c)(x_1 - c)}{2}$.

This shows that $f(x_1) > f(c)$ for all small positive x_1 which is a contradiction.

Similarly, if $f'(c) < 0$, then for all x_1 with $c < x_1 < c + \delta$, we get

$$\frac{f(c) - f(x_1)}{c - x_1} < \frac{f'(0)}{2}$$

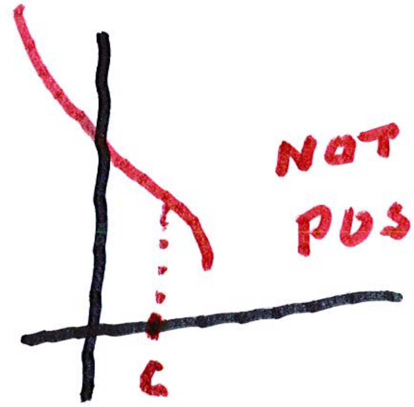
$$f(c) - f(x_1) < \frac{f'(0)(c - x_1)}{2}$$

$$\rightarrow \quad f(x_1) > f(c) + \frac{f'(0)(x_1 - c)}{2}.$$

Thus for all x_1 with

$c - \delta < x_1 < c$, we get

$f(x_1) > f(c)$



NOT POSSIBLE.

Thus f cannot have a relative maximum at c if $f'(c) > 0$ or $f'(c) < 0$.

We get a similar result if f has a relative minimum.

Corollary. Let $f: I \to \mathbb{R}$ be continuous on I. Then either f has a relative extremum or the derivative does not exist at c.

Rolle's Thm.

Suppose that f is continuous on a closed interval $I = [a, b]$, that the derivative f' exists at every point of (a, b), and that $f(a) = f(b) = 0$. Then there is at least one point c in (a, b) where $f'(c) = 0$.

Proof: If $f(x) = 0$, all x.

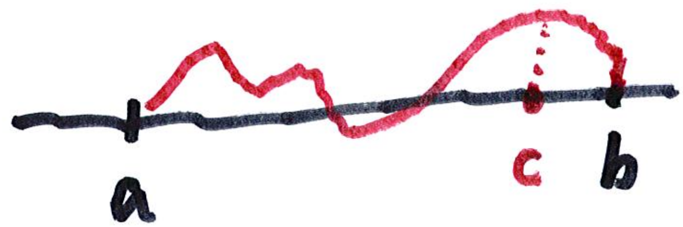
then set $c =$ any $x \rightarrow f'(c) = 0$.

First, we assume f has positive values. By the Maximum-Minimum Theorem, there is at least one point c in $[a, b]$ with an absolute maximum at c.

Clearly c cannot equal a or b. Since f is differentiable at every interior

point, we conclude that

$f'(c) = 0$



Note that if f has no positive

values, then we

can multiply f by (-1),

so that it does have positive

values, Then there is a c

in (a,b) where $f'(c) = 0$.

Mean Value Thm. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point c in (a,b) so that

$$f(b) - f(a) = f'(c)(b-a).$$

Pf. Consider the function

Proof: Consider the function

$$\varphi(x) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

It's easy to verify that

$\varphi(a) = 0$ and $\varphi(b) = 0$,

and also that f is differentiable in (a,b) and continuous in $[a,b]$.

Hence Rolle's Thm implies

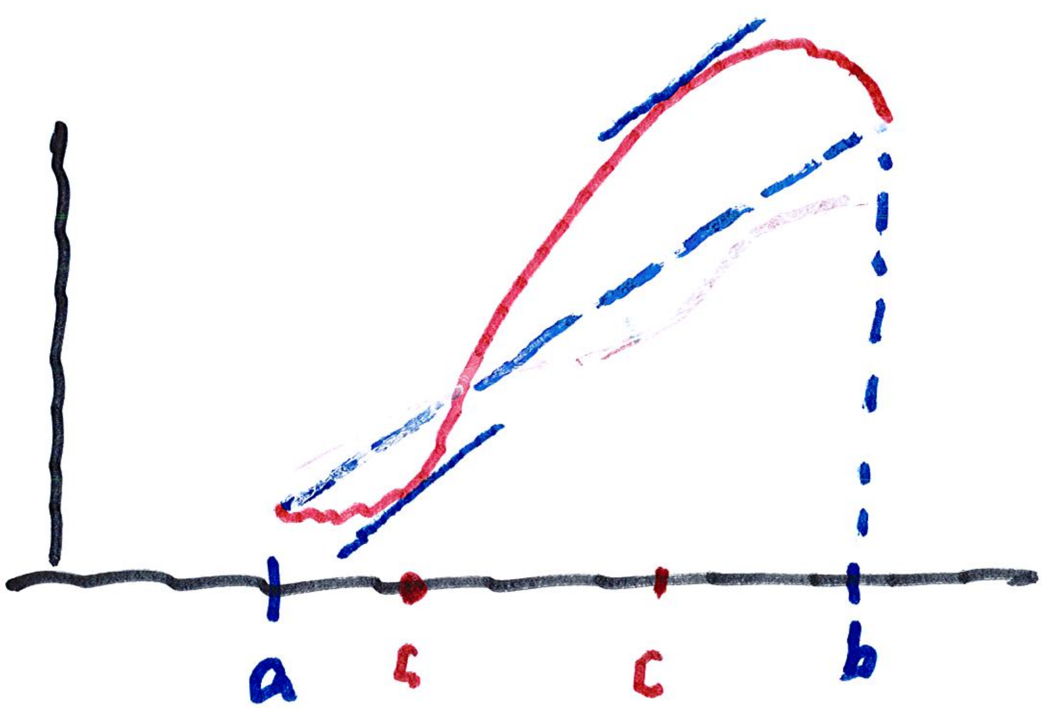
there is a point where

$\varphi'(c) = 0$, i.e. that

satisfies

$$0 = \phi'(c) = \frac{f(b) - f(a)}{b - a}.$$

The fraction on the right is the slope of the line from $(a, f(a))$ to $(b, f(b))$.



Some important theorems:

Suppose that f is continuous on $I = [a, b]$ and that

$f'(x) = 0$ for all $x \in (a, b)$.

Then there is a constant C such that $f(x) = C$, all x in I.

Pf. We show $f(x) = f(a)$ for all x in $[a, b]$. Choose any x between a and b.

The Mean Value Thm. implies

that there is a c in (a, b)

such that

$$f(x) - f(a) = f'(c)(x - a).$$

Since $f'(c) = 0$, all c,

it follows that $f(x) - f(a) = 0$,

for all x.

Thm. Suppose f and g are

continuous on $[a, b]$ and that

$f'(x) = g'(x)$ for all x in (a, b).

Then $\big(f(x) - g(x)\big)' = 0$, all x.

$\therefore\ f(x) - g(x) = C$.

$\Rightarrow\ f(x) = g(x) + C$

Def'n. We say a function f on I is increasing if whenever $x_1 < x_2$, then $f(x_1) \leq f(x_2)$.

Suppose that $f'(x) \geq 0$

for all $x \in I$. Then

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$
$$\geq 0.$$

$\therefore$ f is increasing.

On the other hand, suppose ~~that when $x_1 < x_2$ we have $f(x_1) < f(x_2)$, and that f is decreasing. Then~~

For the converse, we suppose that f is differentiable and increasing on I. Thus, for any $x \neq c$ in I, we have

$$\frac{(f(x) - f(c))}{x-c} \geq 0 . \quad \left\{ \begin{array}{l} \text{because} \\ f \text{ is increasing} \end{array} \right.$$

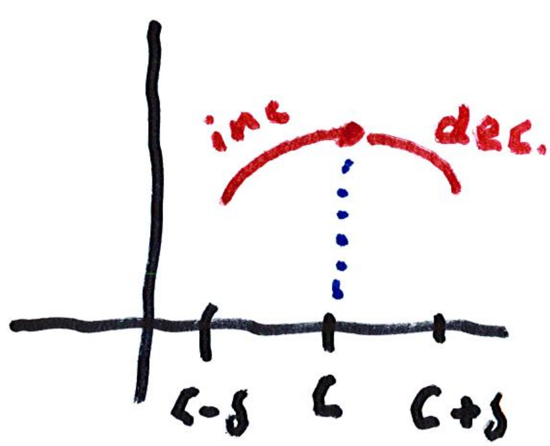
Hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x-c} \geq 0$$

$\therefore$ f increasing on $I \Rightarrow f' \geq 0$ on I.

Thm. First Derivative Test for Extrema.

(a) Suppose $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for x with $c < x < c+\delta$. Then f has a relative maximum at c.



(b) Suppose $f'(x) \leq 0$ for $c - \delta < x < c$ and that $f'(x) \geq 0$ for $c < x < c + \delta$. Then f has a relative minimum at c.