6.3 L'Hopital's Rules

We need to prove a generalization of the Mean Value Thm.

Cauchy Mean Value Theorem

Let f and g be continuous on $I = [a, b]$ and differentiable on (a, b)

Assume that $g'(x) \neq 0$ for all x in (a, b). Then there exists c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(When $g(x) = x$, this is the usual Mean Value Thm.)

Proof. Note that Rolle's Thm implies that $g(a) \neq g(b)$, for if $g(a) = g(b)$, then

$$g'(c) = \frac{g(b) - g(a)}{b-a} = 0,$$

(Contradiction).

Set

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)} \{g(x) - g(a)\} - (f(x) - f(a)).$$

Note that h is continuous on $[a, b]$ and differentiable on (a, b). Then Rolle's Thm that there is in (a, b) so

$$0 = h'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) - f'(c).$$

If we divide by $g'(c)$, we get the desired formula.

There are several versions

of L'Hopital's Rules.

The most common is that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

provided that $f(c) = 0 = g(c)$

and that the usual continuity

and differentiability rules hold.

Today, we'll prove:

L'Hopital's Rule I.

Let $a < b$ and let f, g be differentiable on (a, b).

Assume that $g'(x) \neq 0$ on (a, b) and that

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x).$$

(a) If $\displaystyle\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$,

then $\displaystyle\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Pf. We will arrange the numbers as follows:

$a, \alpha, u, \beta, c, b.$

Cauchy's Mean Val Thm states:

Given $a < \alpha < \beta < b$,

then there is a u with

$\alpha < u < \beta$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(u)}{g'(u)} \qquad (1)$$

If $L \in \mathbb{R}$ is the number

in (a), then

for any $\varepsilon > 0$, there exists

$c \in (a, b)$ such that

$$L - \varepsilon < \frac{f'(u)}{g'(u)} < L + \varepsilon$$

for $u \in (a, c)$.

It follows from (1) that

$$(2) \quad L - \varepsilon < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < L + \varepsilon$$

for $a < \alpha < \beta < c$.

Recall that $\lim_{x \to a^+} f(x) = 0$

and that $\lim_{x \to a^+} g(x) = 0$.

If we take the limit in (2)

as $\alpha \to a^+$, we have

$$L - \varepsilon \le \frac{f(\beta)}{g(\beta)} \le L + \varepsilon$$

for $\beta \in (a, c)$

Since $\varepsilon > 0$ and β is in (a, c).

it follows that if we set $x = \beta$,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Case (b).

If $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L = \infty$

and if $M > 0$ is given,

then

there exists $c \in (a, b)$ such that

$$\frac{f'(u)}{g'(u)} > M \quad \text{for } u \in (a, c)$$

which by (1) implies that

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} > M \quad \text{for}$$

$a < \alpha < \beta < c$

Recall $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$.

Hence, we have (by letting $\alpha \to a$)

$$\frac{f(\beta)}{g(\beta)} \geq M \qquad \text{for } \beta \in (a, c).$$

Since M is arbitray, the assertion follows.