7.1 Riemann Integral (cont'd)

A function $f: [a, b] \to \mathbb{R}$

is said to be Riemann

integrable on $[a, b]$ if

there is a number L in $\mathbb{R}$

such that for every $\varepsilon > 0$

there is a $\delta_\varepsilon > 0$ such that

if $\dot{P}$ is any tagged

partition of $[a, b]$ with

$\| \dot{P} \| < \delta_\varepsilon$, then

$$|S(f; \dot{P}) - L| < \varepsilon. \quad (1)$$

We say that $\int_a^b f = L$,

and we say that $f \in \mathcal{R}[a, b]$.

Note that (1) has to hold

for <u>every</u> $\dot{P}$ with $\| \dot{P} \| < \varepsilon$.

We showed that if f is a constant function k, then

$$\int_a^b k = k(b-a).$$

We studied $\displaystyle\int_0^3 g$.

where $g(x) = \begin{cases} 2, & \text{if } 0 \le x \le 1 \\ 3, & \text{if } 1 < x \le 3 \end{cases}$

If the partition P is

$0 = x_0 < x_1 \ldots, x_n = 3,$

we defined j to be the

larjest integer such that

$x_j \leq 1$. Note this implies

that $x_{j+1} > 1$.

If $x_j < 1$ and if

$t_{j+1} \in [x_j, 1]$, then $g(t_{j+1}) = 2$

and if $t_{j+1} \in (1, x_{j+1}]$,

then $g(t_{j+1}) = 3$

Also, if $x_j = 1$ and if $t_{j+1} = x_j$,

then $g(t_{j+1}) = 2$

and if $x_j < t_{j+1}$, then $g(t_{j+1}) = 3$

The above 4 cases show that

$g(t_{j+1}) = 2$ or 3.

Now we estimate $S(g; \dot{P})$ from above and below:

Above (by telescoping),

$$S(g;\dot{P}) = 2x_j + 3(x_{j+1} - x_j) + 3(x_n - x_{j+1})$$

$$= 9 - x_j \quad (\text{since } x_n = 3)$$

$$S(g;\dot{P}) = 9 - x_{j+1} + (x_{j+1} - x_j)$$

$$\leq 9 - 1 + (x_{j+1} - x_j)$$

Since $x_{j+1} - x_j < \delta$, we get

$$S(g;\dot{P}) = 8 + \delta.$$

From below:

$$S(g;\dot{P}) = 2x_j + 2(x_{j+1} - x_j)$$

$$+ (9 - 3x_{j+1})$$

$$\therefore S(g;\dot{P}) = 9 - x_{j+1}$$

$$= 9 - x_j + (x_j - x_{j+1})$$

$$\geq 8 + (x_j - x_{j+1})$$

$$\therefore S(g;\dot{P}) \geq 8 - \delta;$$

Thus, we've shown that

$$8 - \delta \leq S(g;\dot{P}) \leq 8 + \delta$$

If we let $\delta \to 0$, it follows that $\int_0^3 g = 8$.

~~This implies $|S(g;\dot{P}) - 8| < 3\delta$~~

~~If we set $\delta = \frac{\varepsilon}{3}$.~~

~~$\Rightarrow$ if $\|\dot{P}\| < \delta$, then~~

~~$$|S(g;\dot{P}) - 8| < \varepsilon.$$~~

Ex. 3. Compute $\int_0^1 x \, dx$.

Let $\dot{Q}$ be the partition

$\{x_0, x_1, \ldots, x_N\}$ with the

tag defined by $t_i = q_i = \dfrac{x_i + x_{i-1}}{2}$.

Then $h(x) \equiv x$ satisfies

$$h(q_i)(x_i - x_{i-1}) = \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1})$$

$$= \frac{1}{2}(x_i^2 - x_{i-1}^2).$$

This sum telescopes:

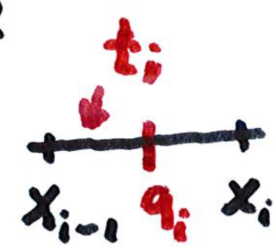
$$S(h;\dot{Q}) = \sum_{i=1}^{n} \frac{1}{2}(x_i^2 - x_{i-1}^2)$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}.$$

Now let $\dot{P}$ be an arbitrary partition of $[0,1]$ with $\|\dot{P}\| < \delta$.

We use q_i = midpoint of I_i.

Note that $|t_i - q_i| < \frac{\delta}{2}$



Using the Triangle Inequality

$$|S(h;\dot{P}) - S(h;Q)|$$

$$= \left| \sum_{i=1}^{n} t_i (x_i - x_{i-1}) - \sum_{i=1}^{n} q_i (x_i - x_{i-1}) \right|$$

$$\leq \sum_{i=1}^{n} |t_i - q_i| (x_i - x_{i-1})$$

$$< \frac{\delta}{2} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\delta}{2}(1-0) = \frac{\delta}{2}.$$

Since $S(h;\dot{Q}) = \frac{1}{2}$, we conclude that if $\|\dot{P}\| < \delta$, then

$$|S(h;\dot{P}) - \tfrac{1}{2}| < \frac{\delta}{2}.$$

Letting $\delta \to 0$, it follows

that $\int_0^1 x = \frac{1}{2}$

Some Properties :

Thm. Suppose that f and g are in $R[a,b]$. Then

(a) If $k \in \mathbb{R}$, then $kf \in R[a,b]$

(b) $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_b^c g$

(c) $f+g \in R[a,b]$, and

$\displaystyle\int f+g = \int$

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g .$$

(c) If $f(x) \leq g(x)$, for all

x in $[a, b]$ and

$$\int_a^b f \leq \int_a^b g .$$

Pf. If $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$

is a tagged partition of $[a,b]$,

then one can show that

$$S(kf;\dot{P}) = k\,S(f;\dot{P})$$

$$S(f+g;P) = S(f;\dot{P}) + S(g;\dot{P})$$

$$S(f;\dot{P}) \le S(g;\dot{P})$$

are easy.

The assertion in (a) follows easily from the first equality

Given $\varepsilon > 0$, one can find $\delta_\varepsilon > 0$ so that

(2) $$\left| S(f; \dot{P}) - \int_a^b f \right| < \frac{\varepsilon}{2} \quad \text{and}$$

$$\left| S(g; \dot{P}) - \int_a^b g \right| < \frac{\varepsilon}{2}$$

To prove (b),

$$\left| S(f+g) - \left(\int_a^b f+g \right) \right|$$

$$= \left| \left(S(f;\dot{P}) - \int_a^b f \right) + \left(S(g;\dot{P}) - \int_a^b g \right) \right|$$

$$\leq \left| S(f;\dot{P}) - \int_a^b f \right| + \left| S(g;\dot{P}) - \int_a^b g \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary,

(b) follows.

To prove (c) {using (2)}

$$\int_a^b f - \frac{\varepsilon}{2} < S(f, \dot{P}) \quad \text{and}$$

$$S(g; \dot{P}) < \int_a^b g + \frac{\varepsilon}{2} ,$$

we get

$$\int_a^b f - \frac{\varepsilon}{2} < S(f, \dot{P})$$

$$\leq S(g; \dot{P}) < \int_a^b g + \frac{\varepsilon}{2}$$

We get

$$\int_a^b f < \int_a^b g + \varepsilon .$$

Since ε is arbitrary, we get

$$\int_a^b f \leq \int_a^b g .$$