7.2 cont'd.

We can use the Cauchy Criterion to show that the Dirichlet function

$f(x) = 1$, if $x \in [0,1]$

and $f(x) = 0$ if $x \in (0,1)$ is irrational.

Set $\varepsilon_0 = \frac{1}{2}$. Let $\dot{P}$ be any Partitionl such that every tag is rational, and let

$\dot{Q}$ be any partition of $[0,1]$ with each tag an irrational number. Then

$$S(f;P) = \sum_{i=1}^{n} 1\cdot(x_i - x_{i-1}) = 1-0$$
$$= 1$$

Also

$$S(f;Q) = \sum_{i=1}^{n} 0\cdot(x_i - x_{i-1}) = 0$$

The same formulas are true for any such partional with arbitrarily small norm.

This shows $|S(f;\dot{P}) - S(f;\dot{Q}| = 1$,

it's clear that the Cauchy

Criterion implies $f \notin \mathcal{R}[0,1]$.

Some examples of

integrable functions

Theorem If $f : [a,b] \longrightarrow \mathbb{R}$

is continuous, then $f \in \mathcal{R}[a,b]$

We know that since f is continuous on $[a,b]$, f is uniformly continuous.

$\therefore$ Given $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ so that if $u, v \in [a,b]$ and $|u-v| < \delta_\varepsilon$, then

$$|f(u) - f(v)| < \frac{\varepsilon}{b-a}.$$

Let $P = \{I_i\}_{i=1}^{n}$ be a partition with $\|P\| < \delta_\varepsilon$. Let $u_i \in I_i$ be a point where f attains its minimum value and let $v_i \in I_i$ where f attains its maximum on I_i

Let α_ε be the step fcn. defined by

$$\alpha_\varepsilon(x) = f(u_i) \quad \text{for } x \in [x_{i-1}, x_i)$$

for $i = 1, \dots, n-1$ and $\alpha_\varepsilon(x) = f(u_n)$

for $x \in [x_{n-1}, x_n]$.

Similarly we define ω_ε,

using v_i instead of u_i. Then

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x),$$

for $x \in [a, b]$.

Moreover we have

$$0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon)$$

$$= \sum_{i=1}^{n} (f(v_i) - f(u_i))(x_i - x_{i-1})$$

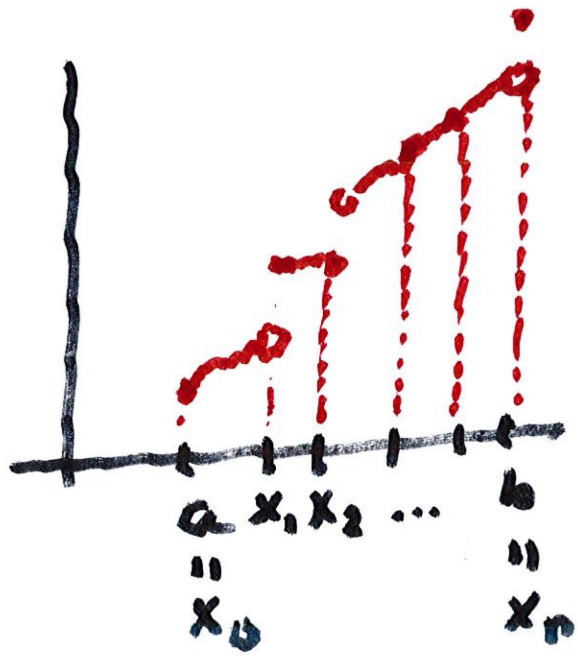
$$\le \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a}\right)(x_i - x_{i-1}) = \varepsilon.$$

The Squeeze Thm implies $f \in R[a,b]$.

Recall a function f is monotone on $[a,b]$ if

$x' < x'' \longrightarrow f(x') < f(x'')$.

Thm. If $f:[a,b]\to\mathbb{R}$ is monotone, then $f\in R[a,b]$.



Pf. We partition $[a,b]$ into n equal subintervals. with

$$x_k - x_{k-1} = \frac{(b-a)}{n}.$$

We define a step function

$\alpha(x) = f(x_{k-1})$ and

$\omega(x) = f(x_k)$ for $x \in [x_{k-1}, x_k)$

and

$\alpha(x) = f(x_{n-1})$ and

$\omega(x) = f(x_n)$ for $x \in [x_{n-1}, x_n]$.

Then $\alpha(x) \leq f(x) \leq \omega(x)$

for all $x \in I$, and

$$\int_a^b \alpha = \frac{b-a}{n}\left(f(x_0) + f(x_1) + \dots + f(x_{n-1}) \right)$$

$$\int_a^b \omega = \frac{b-a}{n}\left(f(x_1) + \dots + f(x_{n-1}) + f(x_n) \right)$$

By subtracting, we get

$$\int_a^b (\omega - \alpha) = \frac{b-a}{n}\left(f(x_n) - f(x_0) \right)$$

$$= \frac{b-a}{n}\left(f(b) - f(a) \right).$$

For a given $\varepsilon > 0$, we choose

$$n > \frac{(b-a)}{n}\{f(b) - f(a)\} / \varepsilon .$$

This gives $\int_a^b (w - \alpha) < \varepsilon$

and the Squeeze Theorem

implies $f \in R[a,b]$.

The First Form of the

Fundamental Theorem of

Calculus gives us a tool

7.3. Fundamental Thm. of Calculus, (First Form)

Suppose there is a finite set E in $[a,b]$ and functions $f, F := [a,b] \to \mathbb{R}$ such that

(a) F is continuous on $[a,b]$,

(b) $F'(x) = f$ for all $x \in [a,b]/E$

(c) f belongs to $R[a,b]$.

Then $$\int_a^b f = F(b) - F(a). \tag{1}$$

Proof. We prove the theorem ~~when E~~ where $E = \{a, b\}$.

The general case can be obtained by breaking the interval into a union of a finite number of intervals.

Let $\varepsilon > 0$. Since $f \in R[a,b]$ by (c), there is $\delta_\varepsilon > 0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \delta_\varepsilon$, then

$$\left| S(f, \dot{P}) - \int_a^b f \right| < \varepsilon. \quad (2)$$

If the subintervals are $[x_{i-1}, x_i]$, then the

Mean Value Thm. applied

to F on $[x_{i-1}, x_i]$ implies

that there is $u_i \in (x_{i-1}, x_i)$

such that

$$F(x_i) - F(x_{i-1}) = F'(u_i) \cdot (x_i - x_i^{-1})$$

for $i = 1, \ldots, n$.

If we add these terms,

(and use telescoping), and

use the fact that $F'(u_i) = f(u_i)$

we obtain

$$F(b) - F(a) = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1}) \right)$$

$$= \sum_{i=1}^{n} f(u_i) \, (x_i - x_{i-1}).$$

Now let $\dot{P}_u = \left\{ [x_{i-1}, x_i], u_i \right\}_{i=1}^{n}$

so the sum on the right

equals $S(f, ; \dot{P}_u)$. If

If we substitute

$F(b) - F(a) = S(f, \dot{P}_U)$ into

(1), we get

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon.$$

Since ε is arbitrary, we

conclude that (1) holds

Example: If $F(x) = \frac{1}{3}x^3$

for all $x \in [a,b]$, then

$F'(x) = x^2$ for all $x \in [a,b]$.

Further, $f = F'$ is continuous

so it is in $R[a,b]$.

Therefore the Fundamental

Theorem (with $E = \emptyset$) implies

that

$$\int_a^b x^2\,dx = F(b) - F(a) = \frac{1}{3}(b^3 - a^3).$$