all $n \geq 4$.

Sometimes the Induction Principle can be expressed as follows.

Let S be a subset of $\mathbb{N}$ such that

(1) $P(1)$ is true.

(2) For every $k \in \mathbb{N}$,

if $P(1), \ldots, P(k)$ are all true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

(This is sometimes called the Principle of Strong Induction.)

Ex. Suppose a sequence

$\{x_n\}$ is defined by

$x_1 = 1$, $x_2 = 2$ and

$$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n).$$

Use Strong Induction to

show that

$1 \le x_n \le 2$, all $n \in \mathbb{N}$.

Let $P(n)$ be the statement that $1 \le x_n \le 2$.

Note that $P(1)$ and $P(2)$ both hold by hypothesis.

Now let $k \in \mathbb{N}$ with $k \ge 2$, and suppose that $P(j)$ is true for all $j \le k$.

Then $x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$\leq \frac{1}{2}(2+2) = 2$

by strong induction hypothesis

and

$x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$

$\geq \frac{1}{2}(1+1) = 1$

by strong ind. hypothesis

Hence $1 \le x_{k+1} \le 2$,

which shows that $P(k+1)$

is true. Thus the Strong

Induction Principle

implies that $P(n)$ is

true for all $n \in \mathbb{N}$.

1.3 Finite and Infinite Sets

Let $N_m = \{1, 2, \ldots, m\}$.

1. A set S has m elements if there is a bijection f from N_m onto S

2. A set S is finite if it has m elements (m is unique).

3. S is infinite if it is not finite

4. A set S is <u>denumerable</u> if there is a bijection of N onto S

5. S is <u>countable</u> if it is either finite or denumerable.

6. S is <u>uncountable</u> if it is not countable

Ex. Some examples.

The set $E = \{2n : n \in \mathbb{N}\}$

of even natural numbers

is denumerable.

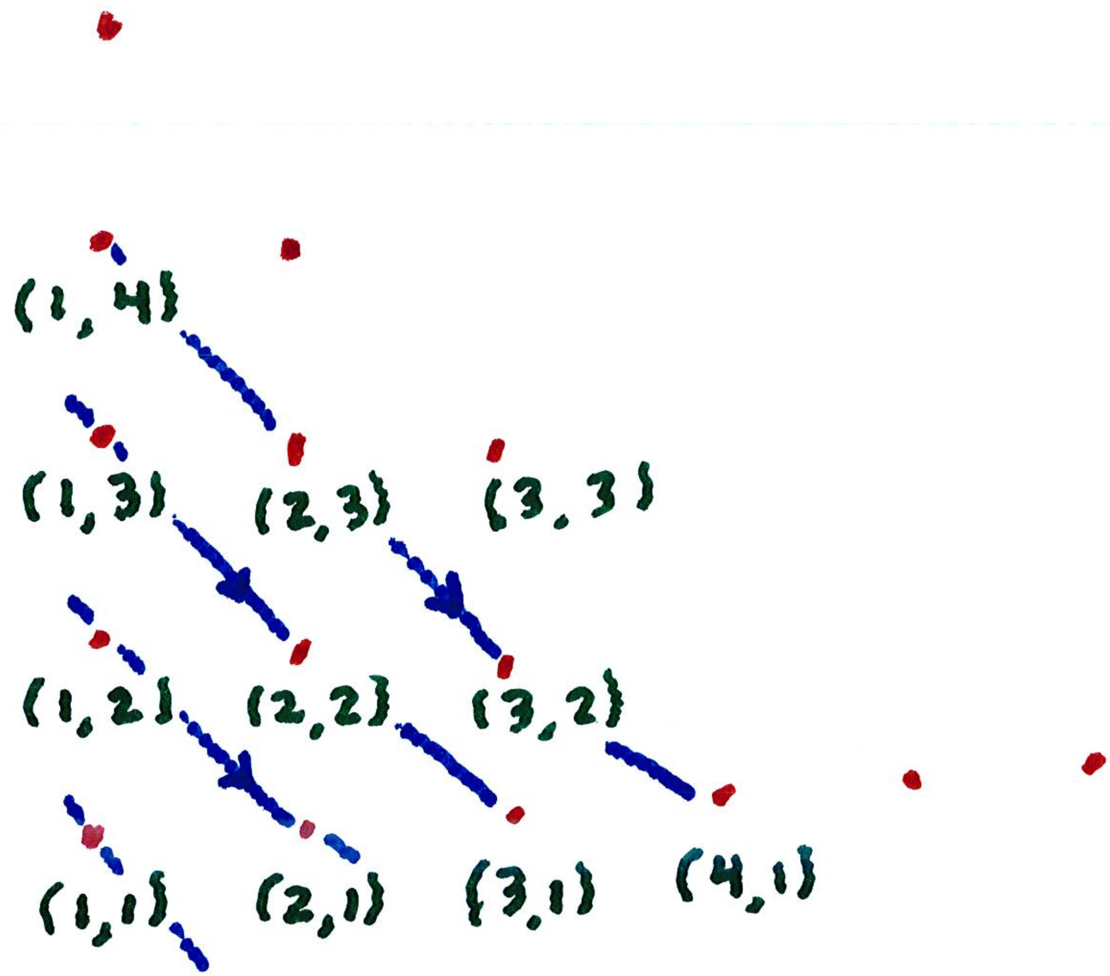
So is $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

So is $P = \{2, 3, 5, 7, 11, \ldots\}$

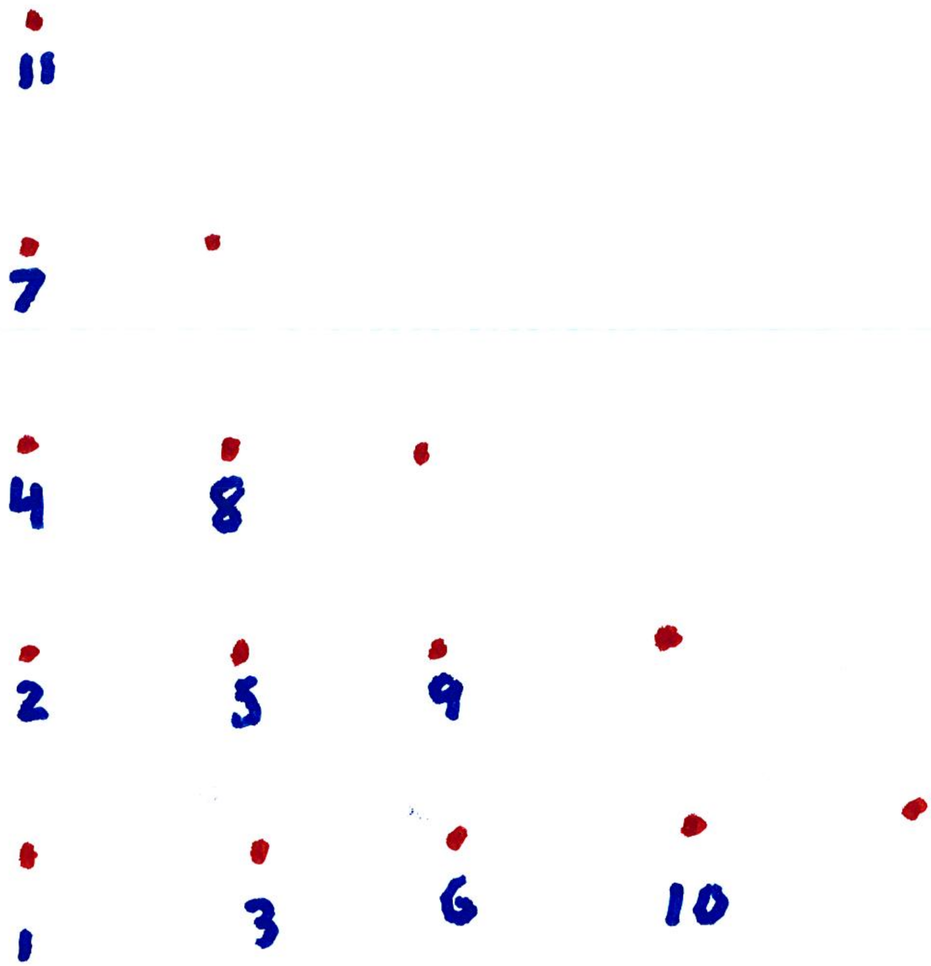
(the set of prime numbers).

$p_1 = 2, \ p_2 = 3, \ p_3 = 5,$ etc.

Is $N \times N$ denumerable?



Follow first diagonal, then the second, then the third, etc. .


11
7
4
8
2
5
9
1
3
6
10

Using this method, let

$f(m,n)$ = value assigned

to (m,n).

Thus $f(1,1) = 1$ $\quad f(1,2) = 2$

$f(2,1) = 3$. $\quad f(1,3) = 4$

... $f(4,1) = 10$, ...

Sum of first 2 diagonals

$= 1 + 2 = 3$ $\quad f(2,1) = 3$

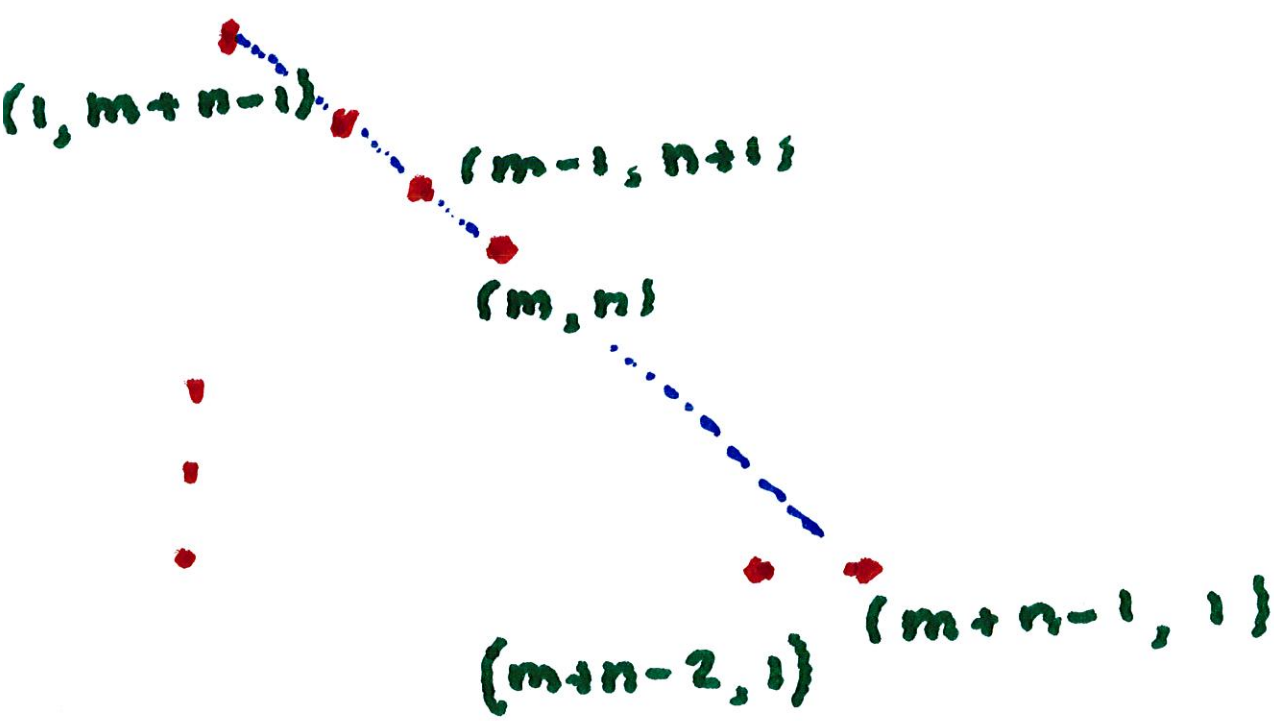
Sum of k diagonals is

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

$$f(k,1) = \frac{k(k+1)}{2}.$$

We see that the endpoints of $(m+n-1)$-th diagonal are $\underline{(1, m+n-1)}$ and $\underline{(m+n-1, 1)}$.

Hence the predecessor of $(1, m+n-1)$ is $(1, m+n-2)$.



Hence,

$$f(m,n) = f(m-1, n+1) + 1$$

$$= f(m-2, n+2) + 2$$

$$\vdots$$

$$= f(1, m+n-1) + (m-1)$$

$$= f(m+n-2, 1) + m$$

$$f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$$

Observe that as we move along the path, $f(m,n)$ increases by 1 with each step. Therefore,

$$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ is 1-to-1 and onto}$$

It follows that f has an inverse $g: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ that is also 1-to-1 and onto.

g satisfies

$g(1) = (1,1)$

$g(2) = (1,2)$

$g(3) = (2,1)$

$g(4) = (1,3)$, etc.

In general

$g(k) = (m(k), n(k))$

for $k = 1, 2, \ldots$

Now define a [illegible]

function $\pi(m, n) = \frac{m}{n}$

and also define

$$h(k) = \pi(g(k)) = \frac{m(k)}{n(k)}$$

This is the k-th positive rational number at the k-th point on the path.

Thus we obtain a

function $h: N \to Q^+$

that is onto but

not 1-to-1.

We want to modify h

to make it 1-to-1 and onto.

But we first prove:

Thm. 1. Suppose that $h : N \to S$ is surjective, where S is infinite. Then there is a function $H : N \to S$ that is 1-to-1 and onto. Thus, S is denumerable.

Proof. Our goal is to

construct a sequence

$n_1, n_2, \ldots, n_\ell, \ldots$

so that all the elements $h(n_i)$

are all distinct, $1 \le i < \infty$

and so every element s of S

equals $h(n_i)$ for some i.

Thus the function $H: N \to S$

defined by

$$H(i) = h(n_i), \quad i = 1, 2, 3, \ldots$$

is a bijection of N onto S.

Set $n_1 = 1$. There are two possible cases:

(i) If $\underline{h(2) \neq h(1) = h(n_1)}$, then set $n_2 = 2$.

or

(ii) There is an integer n_2 with $n_2 \geq 3$, so that $h(n_2) \neq h(n_1)$ and so that $h(k) = h(n_1)$ for all k with $n_1 < k < n_2$.

Similarly, one obtains

a sequence n_ℓ, $\ell = 1, 2, 3, \ldots$ with

$1 = n_1 < n_2 < \ldots < n_\ell < \ldots$

so that all the points

n_ℓ, $\ell = 1, 2, \ldots$ are all distinct

and so that if

$n_{\ell-1} < k < n_\ell$.

then

$$h_k \in \{h_1, h_2, \dots, h_{\ell-1}\}$$

Since all the elements

$h(n_1), h(n_2), \dots, h(n_\ell), \dots$

are distinct, it follows

that $H: N \to S$, defined

by $H(i) = h(n_i)$, $i = 1, 2, \dots$

is 1-to-1.

Also, H is onto since

any $s \in S$ satisfies $s = h(k)$

$= h(n_i)$ for some $n_i < k$.

Hence $s = H(i)$ for some i.

Sets can be arbitrarily large: For any set S, let $\wp(S)$ be the set of all subsets of S.

Cantor's Thm:

There does NOT exis a map $\varphi : S \longrightarrow \wp(S)$ that is onto.

Proof. Suppose

$$\varphi : S \longrightarrow \wp(S)$$

is a surjection.

Since $\varphi(x)$ is a subset of S. Either x belongs to $\varphi(x)$ or it does not belong to $\varphi(x)$. We let

$$D = \left\{ x \in S : x \notin \varphi(x) \right\}$$

Since φ is a surjection,

there exists $x_0 \in S$
such that $\varphi(x_0) = D$.

There are 2 cases:

1. Suppose $x_0 \in D$.

Then $x_0 \in \varphi(x_0)$.

By definition of D,

$x_0 \notin D$. Contradiction

2. Suppose $x_0 \notin D$.

Then $x_0 \notin \varphi(x_0)$.

By definition of D,

$x_0 \in D$. Contradiction.

Ex. Suppose $S = \{a, b, c\}$

$\mathcal{P}(S) = \{ \phi, \{a\}, \{b\}, \{c\},$

$\{a,b\}, \{a,c\}, \{b,c\}$

and $\{a,b,c\} \}$

S has 3 elements,

$\mathcal{P}(S)$ has 8 elements.

There does not exist a surjection from S onto $\mathcal{P}(S)$.