Fundamental Theorem of Calculus, Part 2.

If $f \in R[a,b]$, we define

(1) $$F(z) = \int_a^z f \quad , \qquad \text{for } z \in [a,b]$$

F is called the indefinite integral.

Thm. The indefinite integral defined by (1) is continuous on $[a, b]$. If $|f(x)| \leq M$

on $[a, b]$, then

$$|F(z) - F(w)| \leq M|z-w|,$$

for all z, w in $[a, b]$.

Pf. If $z, w \in [a, b]$ with

$w \leq z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f$$

$$= F(w) + \int_w^z f,$$

which implies

$$F(z) - F(w) = \int_w^z f .$$

Since $-M \leq f(x) \leq M$,
for $x \in [a, b]$,

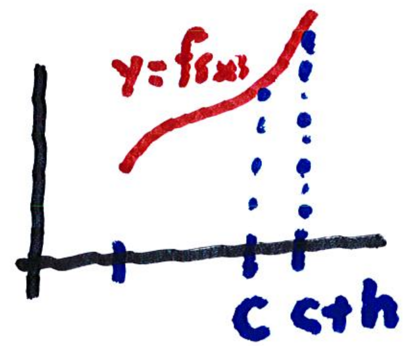
we get

$$-M(z-w) \leq \int_w^z f \leq M(z-w),$$

which gives

$$|F(z) - F(w)| \leq \left| \int_w^z f \right| \leq M|z-w|.$$

We now show F is differentiable at any point where f is continuous



$F(c+h) \approx f(c)h + F(c)$

Fundamental Thm. of Calculus, Part 2. Let $f \in R[a,b]$ and let f be continuous at $c \in [a,b]$.

Then the indefinite integral defined by (1) is differentiable at c and $F'(c) = f(c)$.

Pf. Since f is continuous at c, for any $\varepsilon > 0$ there is $\eta_\varepsilon > 0$ such that if $c \le x < c + \eta_\varepsilon$, then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

The Additivity Theorem

implies that

$$F(c+h) - F(c)$$

$$= \int_a^{c+h} f - \int_a^{c} f = \int_c^{c+} f .$$

If we estimate the above integral for $c \le x \le c+h$, then we get

$$(f(c) - \varepsilon) \cdot h \le F(c+h) - F(c)$$

$$= \int_c^{c+h} f \le (f(c) + \varepsilon) h$$

If we divide by h and subtract $f(c)$, we get

$$-\varepsilon \leq \frac{F(c+h) - F(c)}{h} - f(c) \leq \varepsilon$$

If we let $h \to 0^+$, we obtain

$$-\varepsilon \leq F'(c) - f(c) \leq \varepsilon.$$

Since ε is arbitrary,

we get $F'(c) = f(c)$

Thm. If f is continuous on $[a, b]$, then $F'(x) = f(x)$ for all x in $[a, b]$.

Note that this implies that $F(x)$ (defined by (1)) is an antiderivative, i.e.,

$F'(x) = f(x)$, for all x in $[a, b]$

Ex. If h is Thomae's function, then

$$H(x) = \int_0^x h \text{ is identically } 0$$

on $[0,1]$. However the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \neq h(x)$

when $x \in \mathbb{Q} \cap [0,1]$, so

H is not an antiderivative

of h on $[0,1]$.

We now consider a different

integral that is easier to

compute (called the Darboux integral)

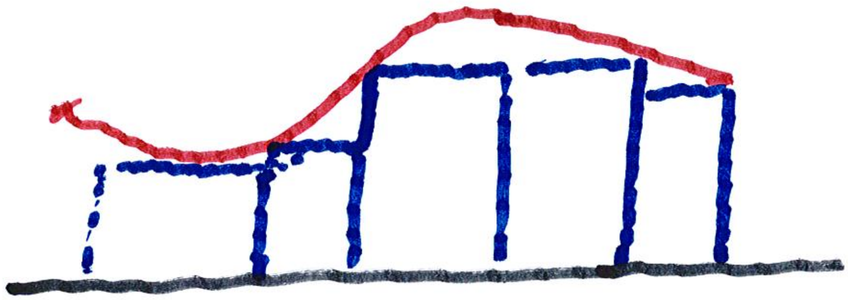
Let $f: I \to \mathbb{R}$ be a bounded function on $I = [a, b]$ and let

$$P = \{x_0, x_1, \dots, x_n\}$$

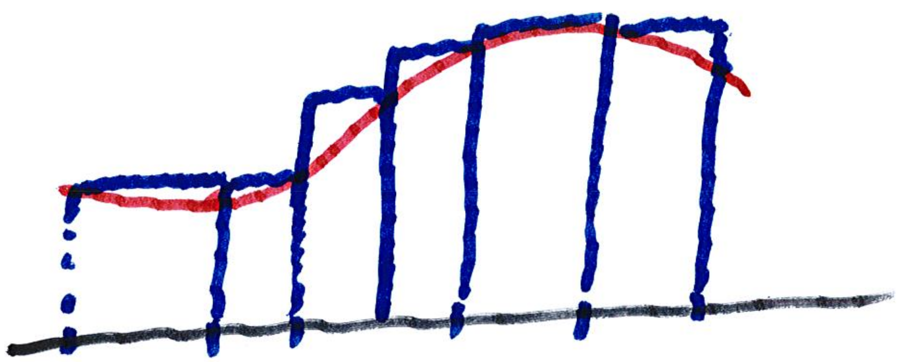
be a partition of I. We let

$$m_k = \inf\{f(x) ; x \in [x_{k-1}, x_k]\}$$

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]$$



$L(f;P)$ lower sum



$U(f;P)$ upper sum

We define a lower sum by

$$L(f;P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

and

and an upper sum by

$$U(f;P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

It is obvious that

$$L(f;P) \leq U(f;P)$$

$\left(\text{since } m_k \leq M_k \quad \text{for } k=1,\dots,n\right)$

Def'n. If P and Q are both partitions of I, then Q is a refinement of P if $P \subset Q$.

Lemma. If Q is a refinement of P, then

$$\left.\begin{aligned} & L(f;P) \le L(f,Q) \\ \text{and } & U(f;Q) \le U(f;P). \end{aligned}\right\} \quad (2)$$

Pf. Suppose Q has just one additional point z that is not in P. We can assume that $Q = \{x_0, \ldots x_{k-1}, z, x_k, \ldots x_n\}$

Let $m_k' = \inf\{f(x); x \in [x_{k-1}, z]\}$

and $m_k'' = \sup\{f(x); x \in [z, x_k]\}$

Then

$$m_k \le m_k' \quad \text{and} \quad m_k \le m_k''$$

Hence

$$m_k(x_k - x_{k-1})$$

$$= m_k(z - x_{k-1}) + m_k(x_k - z)$$

$$\le m_k'(z - x_{k-1}) + m_k''(x_k - z)$$

If we add the terms

$m_j(x_j - x_{j-1})$ for $j \neq k$,

to the above inequality,

we obtain $L(f;P) \leq U(f;Q)$

If Q is any refinement of P,

then we apply the above

result one point at a time

we obtain (2).

The argument for upper sums is the same.

We now show every lower sum is $\leq$ every upper sum:

Lemma. If P_1 and P_2 are two partitions of I, then

$$L(f; P_1) \leq U(f; P_2)$$

Pf. We let $Q = P_1 \cup P_2$, so that

Q is a refinement of both

P_1 and P_2, then

$$L(f;P_1) \le U(f;P_2)$$

Pf.

$$L(f;P_1) \le L(f;Q) \le U(f;Q \le U(f;P_2)$$