Fundamental Theorem of Calculus, Part 2.

If $f \in R[a,b]$, we define

$$F(z) = \int_a^z f \quad , \qquad \text{for } z \in [a,b] \tag{1}$$

F is called the indefinite integral.

Thm. The indefinite integral defined by (1) is continuous on $[a, b]$. If $|f(x)| \le M$

on $[a, b]$, then

$$|F(z) - F(w)| \le M|z-w|,$$

for all z, w in $[a, b]$.

Pf. If $z, w \in [a, b]$ with

$w \le z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f$$

$$= F(w) + \int_w^z f,$$

which implies

$$F(z) - F(w) = \int_w^z f .$$

Since $-M \le f(x) \le M$,

for $x \in [a, b]$,

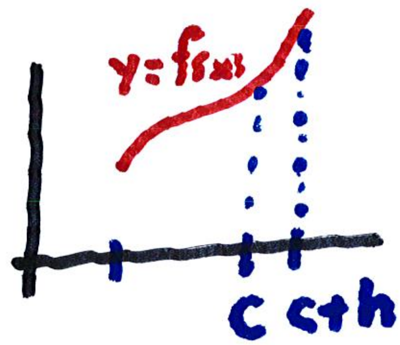
we get

$$-M(z-w) \le \int_w^z f \le M(z-w),$$

which gives

$$\left| F(z) - F(w) \right| \le \left| \int_w^z f \right| \le M |z-w| .$$

We now show F is differentiable at any point where f is continuous



$F(c+h) \approx f(c)h + F(c)$

Fundamental Thm. of Calculus, Part 2. Let $f \in R[a,b]$ and let f be continuous at $c \in [a,b]$.

Then the indefinite integral defined by (1) is differentiable at c and $F'(c) = f(c)$.

Pf. Since f is continuous at c, for any $\varepsilon > 0$ there is $\eta_\varepsilon > 0$ such that if $c \le x < c + \eta_\varepsilon$, then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

The Additivity Theorem

implies that

$$F(c+h) - F(c)$$

$$= \int_a^{c+h} f - \int_a^{c} f = \int_c^{c+h} f.$$

If we estimate the above integral for $c \le x \le c+h$, then we get

$$(f(c) - \varepsilon) \cdot h \le F(c+h) - F(c)$$

$$= \int_c^{c+h} f \le (f(c) + \varepsilon) h$$

If we divide by h and subtract $f(c)$, we get

$$-\varepsilon \le \frac{F(c+h) - F(c)}{h} - f(c) \le \varepsilon$$

If we let $h \to 0^+$, we obtain

$$-\varepsilon \le F'(c) - f(c) \le \varepsilon.$$

Since ε is arbitrary, we get $F'(c) = f(c)$

Thm. If f is continuous on $[a, b]$, then $F'(x) = f(x)$ for all x in $[a, b]$.

Note that this implies that $F(x)$ (defined by (1)) is an antiderivative, i.e.,

$F'(x) = f(x)$, for all x in $[a, b]$

Ex. If h is Thomae's function, then

$$H(x) = \int_0^x h$$ is identically 0 on $[0,1]$. However the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \neq h(x)$

when $x \in \mathbb{Q} \cap [0,1]$, so

H is not an antiderivative

of h on $[0,1]$.

We now consider a different

integral that is easier to

compute (called the Darboux integral)

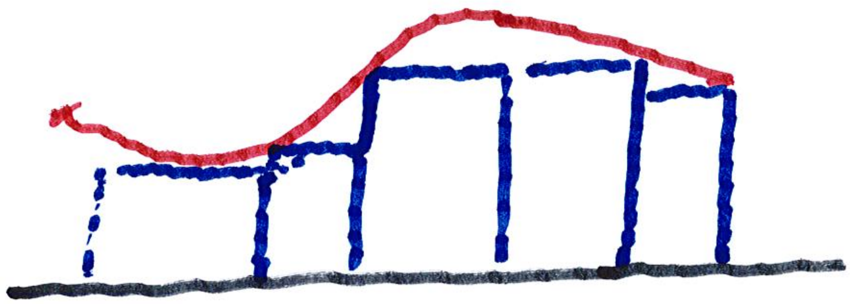
Let $f: I \to \mathbb{R}$ be a bounded function on $I = [a, b]$ and let

$$\mathcal{P} = \{x_0, x_1, \dots, x_n\}$$

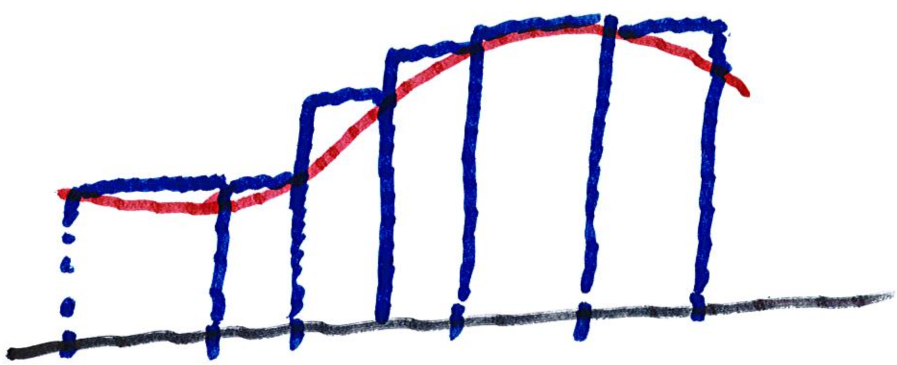
be a partition of I. We let

$$m_k = \inf\{f(x) ; x \in [x_{k-1}, x_k]\}$$

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]$$



$L(f;P)$ lower sum



$U(f;P)$ upper sum

We define a lower sum by

$$L(f;P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

and

and an upper sum by

$$U(f;P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

It is obvious that

$$L(f;P) \leq U(f;P)$$

$\left(\text{since } m_k \leq M_k \text{ for } k=1,\ldots,n\right)$

Def'n. If P and Q are both partitions of I, then Q is a refinement of P if $P \subset Q$.

Lemma. If Q is a refinement of P, then

$$\left.\begin{aligned} L(f;P) &\le L(f,Q) \\ \text{and} \quad U(f;Q) &\le U(f;P). \end{aligned}\right\} \quad (1)$$

Pf. Suppose Q has just one additional point z that is not in P. We can assume that $Q = \{x_0, \dots x_{k-1}, z, x_k, \dots x_n\}$

Let $m_k' = \inf\{f(x);\ x \in [x_{k-1}, z]\}$

and $m_k'' = \sup\{f(x);\ x \in [z, x_k]\}$

Then

$m_k \leq m_k'$ and $m_k \leq m_k''$

Hence

$m_k(x_k - x_{k-1})$

$= m_k(z - x_{k-1}) + m_k(x_k - z)$

$\leq m_k'(z - x_{k-1}) + m_k''(x_k - z)$

If we add the terms

$m_j\,(x_j - x_{j-1})$ for $j \neq k$,

to the above inequality,

we obtain $L(f;\, P) \le U(f;\, Q)$ (2)

If Q is any refinement of P,

then we apply the above

result one point at a time

We obtain (2).

The argument for upper sums is the same.

We now show every lower sum is $\leq$ every upper sum:

Lemma. If P_1 and P_2 are two partitions of I, then

$$L(f; P_1) \leq U(f; P_2)$$

Pf. We let $Q = P_1 \cup P_2$, so that

Q is a refinement of both

P_1 and P_2, then

$$L(f; P_1) \leq U(f; P_2)$$

Pf.

$$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

(3)

Darboux Integral.

Given a bounded function $f: I \longrightarrow \mathbb{R}$, we define the lower integral of f on I by

$$L(f) = \sup \left\{ L(f:P) : P \in \mathcal{P}(I) \right\}$$

where $\mathcal{P}(I)$ is the set of partitions of I. Similarly we define the upper integral by

$$U(f) = \inf\left\{ U(f;P) : P \in \mathcal{P}(I) \right\}.$$

Thm. The lower integral $L(f)$ and the upper integral $U(f)$ on I both exist.

Moreover $L(f) \le U(f)$. (4)

If P_1 and P_2 are any pair of partitions of I, then

then it follows that

$$L(f: P_1) \leq U(f; P_2).$$

$\therefore$ the number $U(f: P_2)$ is

an upper bounded for

the set $\{L(f: P);\ P \in \mathcal{P}(I)\}$

Hence, $L(f)$, being the

supremum of the set satisfies

$$L(f) \leq U(f: P_2).$$

Since P_2 is an arbitrary partition of I, then $L(f)$ is a lower bound for the set $\{U(f:P) : P \in \mathcal{P}(I)\}$.

Hence the infimum of this set set satisfies $L(f) \leq U(f)$.

Def'n. Let $f: I \to \mathbb{R}$ be a bounded function I. We say

f is Darboux integrable on I if $L(f) = U(f) = \int_a^b f$

Ex. Remember how hard it was to calculate $\int_0^3 g$ for the function $g(x) = \begin{cases} 2 & \text{if } 0 \le x \le 1 \\ 3 & \text{if } 1 < x \le 3 \end{cases}$

For $\varepsilon > 0$, we define

$P_\varepsilon = (0, 1, 1+\varepsilon, 3)$. We get the upper sum

$$U(g : P_\varepsilon) = 2 \cdot (1-0) + 3(1+\varepsilon-1)$$

$$+ 3(2-\varepsilon)$$

$$= 2 + 3\varepsilon + 6 - 3\varepsilon = 8.$$

Therefore, $U(g) \leq 8$.

(Recall $U(g)$ is the infimum of

<u>all</u> partitions of $[0,3]$.)

Similarly the lower sum is

$$L(g: P_\varepsilon) = 2 + 2\varepsilon + 3(2-\varepsilon) = 8-\varepsilon$$

so that $L(g) \geq 8$. Then

$$8 \leq L(g) \leq U(g) = 8.$$

which means $L(g) = U(g) = 8$

$\therefore$ The Darboux integral of g is $\int_0^3 g = 8$.

Integrability Criterion.

Let $I = [a, b]$ and let

$f: I \longrightarrow \mathbb{R}$ be a bounded fcn.

on I. Then f is Darboux

integrable if and only if

for each $\varepsilon > 0$, there is a

partition P_ε of I such that

$$U(f : P_\varepsilon) - L(f; P_\varepsilon) < \varepsilon. \quad (5)$$

Pf. If f is integrable, then we have $L(f) = U(f)$. If $\varepsilon > 0$ then since the lower integral is a supremum, there is a partition P_1 of I such that

$$L(f) - \frac{\varepsilon}{2} < L(f; P_1).$$

Similarly there is a partition

P_2 of I such that

$$U(f; P_2) < U(f) + \frac{\varepsilon}{2}.$$

If we let $P_\varepsilon = P_1 \cup P_2$, then

P_ε is a refinement of

P_1 and P_2. Hence

$$L(f) - \frac{\varepsilon}{2} < L(f; P_1) \le L(f: P_\varepsilon)$$

$$\le U(f; P_\varepsilon) \le U(f: P_2) < U(f) + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad U(f:P_\varepsilon) < U(f) + \frac{\varepsilon}{2} \quad \text{and}$$

$$-L(f;P_\varepsilon) < -L(f) + \frac{\varepsilon}{2}$$

Adding together and using $U(f) = L(f)$,

$$U(f:P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

For the converse, note that

$L(f:P) \leq L(f)$ and

$U(f) \leq U(f;P_\varepsilon)$.

Hence

$$U(f) - L(f) \leq U(f;P) - L(f;P)$$

Now for each $\varepsilon > 0$, suppose there is a partition P_ε such that (5) holds. Then we have

$$U(f) - L(f) \leq \varepsilon.$$

Since ε is arbitrary, we conclude $U(f) \leq L(f)$. But we have $L(f) \leq U(f)$ is always true, so we have

$$U(f) - L(f) \leq 0$$ and

$$U(f) - L(f) \geq 0 .$$

It follows $U(f) = L(f)$,

so f is Darboux integrable