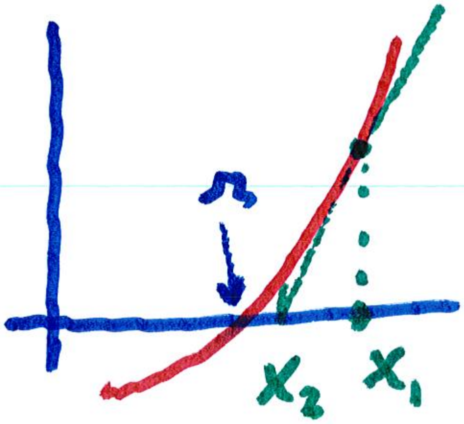
6.4.7 Newton's Method



Assume curve is a straight line:

Solve for r:

$$f(x_1) + f'(x_1)(r - x_1) = 0$$

Divide by $f'(x_1)$:

$$\frac{f(x_1)}{f'(x_1)} = x_1 - r$$

$$\rightarrow r = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Set $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$\vdots$$

Assuming x_n has been found,

$$x_{n+1} = x_n - \frac{f(x_1)}{f'(x_1)}$$

If the initial point x_1 is not toofar from r, the sequence (x_n) converges very rapidly to r.

Thm. Let $I = [a, b]$ and

1. let $f: I \to \mathbb{R}$ be twice differentiable, and

2. Suppose that $f(a) f(b) < 0$.

3. Assume there are constants m and M such that

$|f'(x)| \geq m > 0$ and

$|f''(x)| \leq M$. Let $K = \dfrac{M}{2m}$.

Then there is a subinterval I^* containing a zero r of f such that for any $x_1 \in I^*$, the sequence (x_n) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in N$$

belongs to I^* and converges to r. In fact

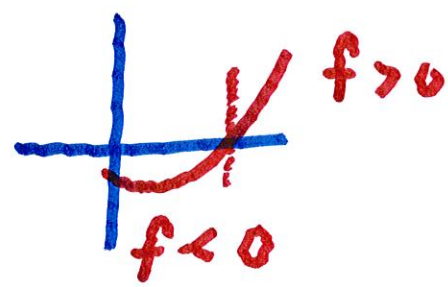
$$|x_{n+1} - r| \leq K |x_n - r|^2 .$$

Pf. $f(a)f(b) < 0$, so $f(a)$ and $f(b)$ have opposite signs.

Also, since $f' \neq 0$ on I,

so there is a single point r

where $f(r) = 0$



Let $x' \in I$ be arbitrary.

Using Taylor's Thm. there is a point c' between x' and r such that

$$0 = f(r) = f(x') + f'(x')(r - x') + \frac{1}{2} f''(c')(r - x')^2,$$

which implies

$$-f(x') = f'(x')(r - x') + \frac{1}{2} f''(c)(r - x')^2$$

If x'' is the number defined by Newton's Procedure:

$$x'' = x' - \frac{f(x')}{f'(x')},$$

then a calculation implies

$$-\frac{f(x')}{f'(x')} = r - x' + \frac{1}{2}\frac{f''(c')(r-x')^2}{f'(x')}$$

or

$$x'' = x' - \frac{f(x')}{f'(x')} = + r + \frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2$$

which implies

$$x'' - r = \frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2 .$$

Using the bounds for $|f'|$ and $|f''|$ and setting $K = \frac{M}{2m}$, we obtain

$$|x'' - r| \le K |x' - r|^2 \qquad (1)$$

We now choose $\delta > 0$

so small so that $\delta < \frac{1}{K}$,

and that the interval

$I^* = [r - \delta, r + \delta]$ is

contained in I. If

$x_n \in I^*$, then $|x_n - r| \le \delta$

Hence (1) implies

$$|x_{n+1} - r| \le K\,|x_n - r|^2$$

$$\le K\delta^2 < \delta$$

Hence $x_n \in I^*$ implies that

$x_{n+1} \in I^*$ for all $n \in \mathbb{N}$.

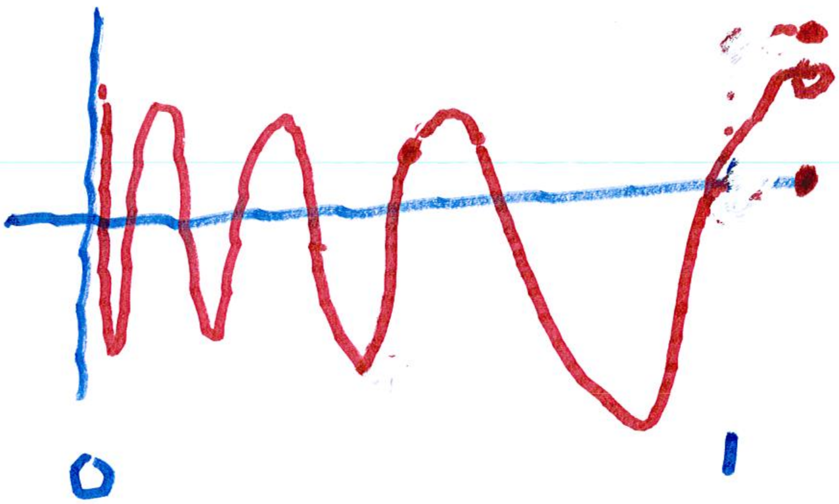
By induction, one can

show that $|x_{n+1} - r| < K\delta^n\,|x_1 - r|$

for all n; Thus $\lim x_n = r$.

The method may not converge if the initial guess is too far off.

Thm. Suppose that $f(x)$ is a bounded function on $[a, b]$ and continuous at all points except for a finite set $c_1, c_2, \dots, c_N$. Then f is in $R[a, b]$. We can do this by assuming that f is continuous at all points except a and b.



Suppose $|f(x)| \le M$, all x in $[a, b]$

Let $\varepsilon > 0$.

Set $\delta' = \dfrac{\varepsilon}{8M}$. Note that

Since f is continuous on

$I = [a+\delta', b-\delta']$, f is uniformly

continuous on I. Thus there

is a small constant $\delta > 0$

so that if $|x' - x''| < \delta$

and if $x', x'' \in [a+\delta, b-\delta]$,

then $|f(x') - f(x'')| < \dfrac{\varepsilon}{16(b-a)}$

Now choose a partition P with

$$a + \delta' = x_0 < x_1 \ldots < x_N = b - \delta',$$

where $\mathrm{Max}\{|x_k - x_{k-1}| ; 1 \le k \le N\}$

is $< \delta$.

Note that if $x \in [x_{k-1}, x_k)$,

then and $x_{N-1} \le x \le x_N$

$$f(x_k) - \frac{\varepsilon}{16(b-a)} \le f(x) + \frac{\varepsilon}{16(b-a)},$$

Now set $M_k' = f(x_k) + \dfrac{\varepsilon}{16(b-a)}$

and set $m_k' = f(x_{k-1}) - \dfrac{\varepsilon}{16(b-a)}$,

then the upper sum $U(f:P)$

is $\leq \dfrac{M\delta}{8} + \sum_{k=1}^{N} \left(f(x_k) + \dfrac{\varepsilon}{(b-a)}\right)(x_k - x_{k-1})$

$+ \dfrac{M\delta}{8}$

Similarly the lower sum $L(f:P)$

is bounded above

$$L(f:P) \geq -M\delta/8 + \sum_{k=1}^{N} \left(f(x_k) - \frac{\varepsilon}{(\ )}\right)(x_k - x_{k-1}) - M\delta/8$$

Hence

$$U(f:P) - L(f:P) \leq \frac{4M\delta}{8} + \sum_{k=1}^{N} 2f(x_k)(x_k - x_{k-1}) + \sum_{k=1}^{N} \frac{2\varepsilon}{(b-a)}(x_k - x_{k-1})$$

$$\leq \frac{4M\delta}{8} + \frac{2\varepsilon}{16(b-a)}$$