Some Comments about the Proof.

1. If $Q_n(x) = c_n(1-x^2)^n$

and $|x|^2 \geq \delta^2$, then

$Q_n(x) \leq \sqrt{n}\,(1-\delta^2)^2$

2. If $P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt.$

when $x+t = 0 \rightarrow t = -x$

when $x+t = 1 \rightarrow t = 1-x$

$\therefore$ Integral for $P_n(x)$ is

$$P_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt$$

3. The change of variables

in the integral is

$\left\{ s = x+t \rightarrow ds = dt \right\}$

$$\int_0^1 f(s)\, Q_n(s-x)\, dx$$

4.

Note that

$$C_n \left(1 - (5-x)^2\right)^n$$

$$= C_n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (5-x)^{2k}$$

$$= \sum_{j+k \leq 2n} d_{j,k}\, 5^j x^k .$$

the integral formula for

$P_n(x)$, one obtains a polynomial

of degree $2n$.

5.

$$P_n(x) - f(x)$$

$$= \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt - \int_{-1}^{1} f(x)\, Q_n(t)\, dt$$

$$= \int_{-1}^{1} \left[f(x+t) - f(x) \right] Q_n(t)\, dt \, .$$

Now we prove our second theorem.

Fundamental Theorem of Algebra. Suppose $a_0, a_1, \ldots, a_n$ are complex numbers, $n \geq 1$, with $a_n \neq 0$. Then

$$P(z) = \sum_{k=0}^{n} a_k z^k = 0 \text{ for}$$

some complex number z_0

Proof. Without loss of generality, assume $a_n = 1$. Put

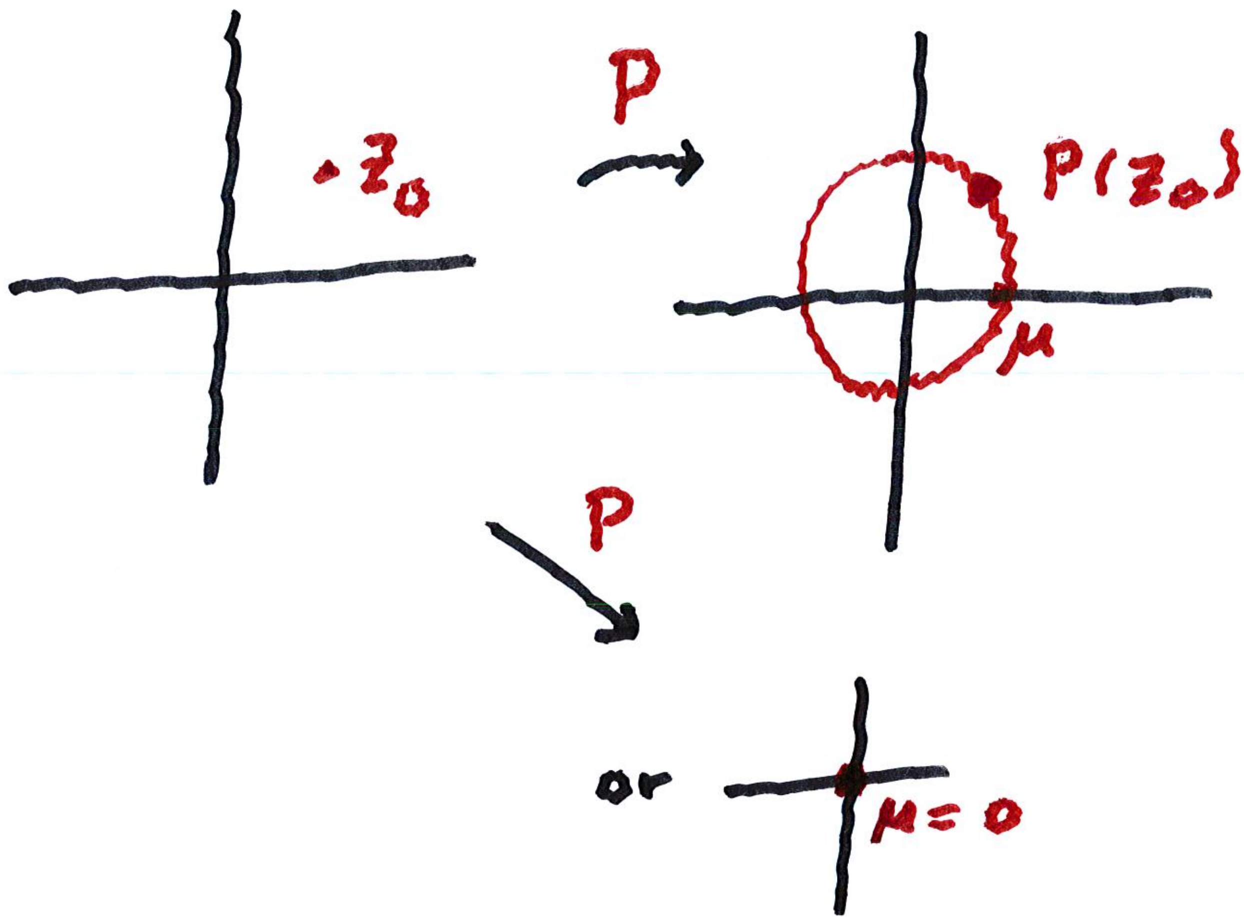
$$\mu = \text{g.l.b.}\ |P(z)|, \qquad (z \text{ complex})$$

If $|z| = R$, then

$$(1) \quad |P(z)| \geq R^n \left[1 - |a_{n-1}| R^{-1} - \cdots - |a_0| R^{-n} \right]$$

The right hand side of (1) tends to ∞ as $R \to \infty$. Hence there exists R_0 such that $|P(z)| > \mu$ if $|z| > R_0$

Since $|P|$ is continuous on the closed disk with center at 0 and radius R_0, the analog of the Maximum-Minimum Thm for the function $|P(z)|$ shows $|P(z_0)| = \mu$ for some z_0.



We claim that $\mu = 0$.

If not, put $Q(z) = \dfrac{P(z+z_0)}{P(z_0)}$

Then Q is a nonconstant

polynomial, $Q(0) = 1$, and

$|Q(z)| \geq 1$ for all z.

There is a smallest integer k, with $1 \leq k \leq n$, such that

$$Q(z) = 1 + b_k z^k + \cdots + b_n z^n,$$

with $b_k \neq 0$.

There is a real θ such that

$$e^{ik\theta} = \frac{-|b_k|}{b_k}, \quad \text{i.e.,}$$

$$e^{ik\theta} b_k = -|b_k| .$$

Then

$$|Q(re^{ik\theta})| \geq 1 - r^k\left[|b_k| - r|b_{k+1}| - \cdots - r^{n-k}|b_n|\right]$$

if $r > 0$.

For sufficiently small r, the expression in braces is positive.

Hence $|Q(re^{i\theta})| < 1$,

which is a contraction.

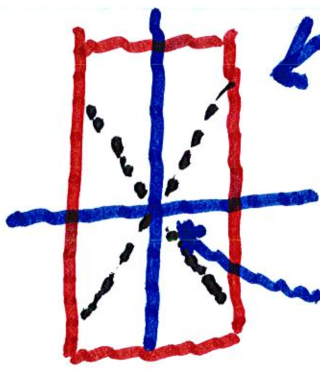
Thus $\mu = 0$, that is, $P(z_0) = 0$.

For our third theorem, we will show how to find the solution of a first order differential equation:

$$\frac{dy}{dx} = f(x, y(x)), \quad y(x_0) = y_0.$$

Suppose that T denotes the rectangular region defined by

$|x - x_0| \leq h$ and $|y - y_0| \leq b$



Suppose that $M = \sup |f(x,y)|$

for all $(x,y) \in T$.

By shrinking h if necessary

we can assume that the

lines $y = y_0 \pm M(x - x_0)$

fit in T for all x, y with

$Mh = b$

We want to find a function

$y(x)$ that is continuously

differentiable and that

satisfies $\frac{dy}{dx}(x) = f(x, y(x))$

with $|x - x_0| \leq h$, and

$y(x_0) = y_0$.

We will find it very useful

for f to satisfy

$$\left| \frac{\partial f}{\partial y}(x,y) \right| \le K. \qquad (1)$$

By the Mean Value Thm,

this means

$$f(x,y_2) - f(x,y_1) = \frac{\partial f}{\partial y}(x,c)$$

for some c.

This means

$$|f(x,y_2)-f(x,y_1)| \le K|y_2-y_1|$$

This is a "Lipschitz Condition"

which we will use instead of

(1).

If a solution to

$y'(x)=f(x)$, for $|x-x_0|\le h$.

exists,

this would imply

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\, dt$$

Setting $y(x_0) = y_0$, this means

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

We solve this by iteration

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt.$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

Note that

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_0)\,dt \right|$$

$$\leq M \left| \int_{x_0}^{x} dt \right|$$

$$= M|x - x_0| \leq Mh.$$

$$\leq b$$

More generally, if we assume $|y_n(x) - y_0| \le Mh \le b$,

then

$$|y_{n+1}(x) - y_0| \le \left| \int_{x_0}^{x} f(t, y(t))\, dt \right|$$

$$\le M|x - x_0| \le b$$

Thus, for all $n = 1, 2, \ldots$

$$|y_n(x) - y_0| \le M|x - x_0|$$

$$\le Mh \le b$$