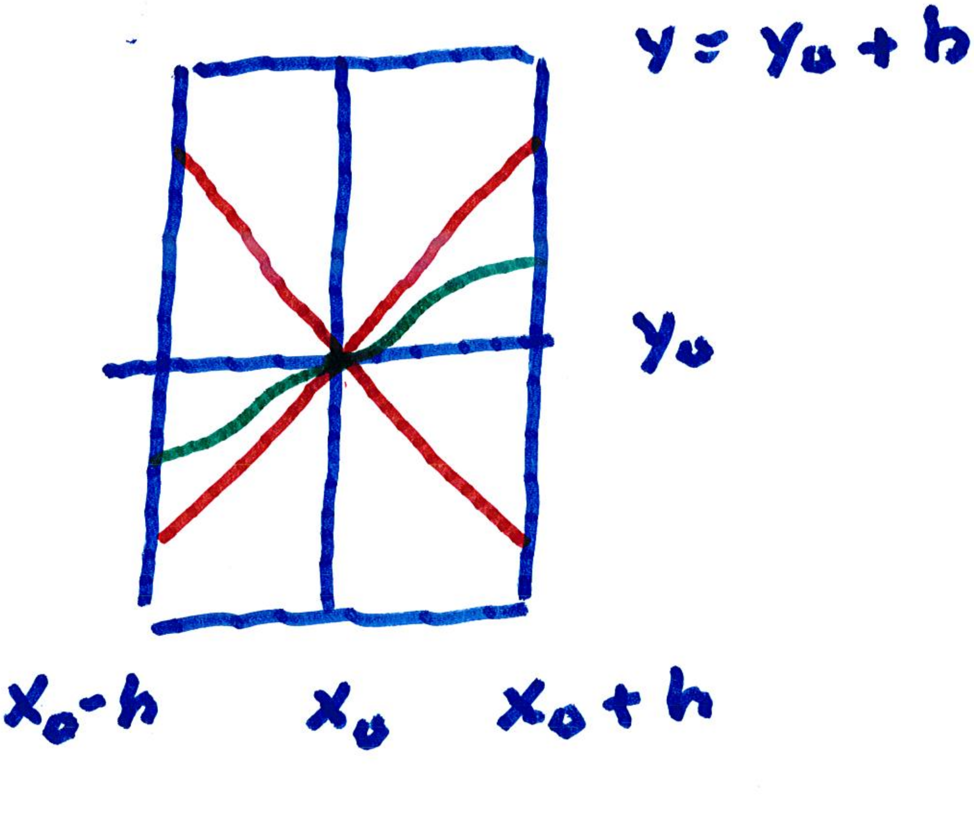
Suppose that $f(x,y)$ is a continuous function on the closed rectangle

$$T = \left\{ (x,y) : \begin{array}{l} |x - x_0| \leq h \\ |y - y_0| \leq b \end{array} \right\}$$

Let $M = \sup \left\{ |f(x,y)| : (x,y) \in T \right\}$

By shrinking h if necessary we can assume that $Mh \leq b$.

Geometrically, this means that the lines $y = y_0 \pm M(x - x_0)$ pass through the vertical lines $x = x_0 - h$ or $x = x_0 + h$



We will show that the curves $y = y(x)$, $y(x_0) = y_0$ will stay in the triangular regions.

Our strategy is to find a sequence of curves by

$$y_0(x) = y_0 \,, \quad \text{when } |x - x_0| \leq h$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0(x))\, dx$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f\{t, y_1(t)\}\, dt$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f\{t, y_{n-1}(t)\}\, dt$$

We hope to show that

$$y_n(x) \longrightarrow y(x), \quad \text{as } n \to \infty,$$

so that

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y$$

if $|x - x_0| \leq h$.

Lemma 1. If $|x - x_0| \le h$, then

$$\left| y_n(x) - y_0 \right| \le M\,|x - x_0| \le Mh \le b.$$

We will do this by Induction

In the case when $n = 1$,

we have

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_0)\,dt \right|$$

$$\le \int_{x_0}^{x} |f(t, y_0)|\,dt$$

Now suppose that for $|x - x_0| \leq h$, we have that $(x, y_k(x))$ is in T so that $|f(x, y_k(x))| \leq M$. Thus,

$$|y_{k+1}(x) - y_0| \leq \left| \int_{x_0}^{x} f(t, y_k(t))\,dt \right|$$

$$\leq M \left| \int_{x_0}^{x} dt \right| \leq M|x - x_0|$$

$$\leq Mh \leq b.$$

This proves the inductive step. We conclude that

$$|y_n(x) - y_0| \le M|x - x_0| \le h.$$

for all $n = 1, 2, \ldots$

Geometrically, this shows that each curve lies in the triangular regions.

We will also need

Lemma 2: If $|x - x_0| \le h$ and $n = 1, 2, \ldots,$ then

$$|f(x, y_n(x)) - f(x, y_{n-1}(x))| \le K\,|y_n(x) - y_{n-1}(x)|.$$

Lemma 3: If $|x - x_0| \le h$, then

$$|y_n(x) - y_{n-1}(x)| \le \frac{M K^{n-1} |x - x_0|^n}{n!} \le \frac{M K^{n-1} h^n}{n!}$$

For $n=1$, from Lemma 1,

$$|y_1(x) - y_0| \leq M\,|x-x_0|.$$

Now the Inductive Step is

$$(1)\quad |y_{n-1}(x) - y_{n-2}(x)| \leq MK^{n-2}\,|x-x_0|^{n-1}$$

We want to show that

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}\,|x-x_0|^n}{n!}$$

We'll do this when $x_0 \leq x \leq x_0+h$.

(the case when $x_0 - h \leq x \leq x_0$ is similar)

From Lemma 2, we have

$$|y_n(x) - y_{n-1}(x)|$$

$$= \left| \int_{x_0}^{x_0+h} [f(t, y_{n-1}(t)) - f(t, y_{n-1})] dt \right|$$

$$\leq \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))| dt$$

$$\leq K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)| dt$$

Using the Inductive Assumption,

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}}{(n-1)!} \int_{x_0}^{x} (t - x_0)^{n-1} dt$$

or

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1} \, |x - x_0|^n}{n!}.$$

This completes the proof of Lemma 3. From Lemma 3, we have two infinite series

$$\sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)]$$ and

$$\sum_{n=1}^{\infty} \frac{MK^{n-1} h^n}{n!}.$$

The second series is an absolutely convergent series and the second series dominates the first (term-by term). The Weierstrass M Test implies that the series $\sum_{k=1}^{\infty} \left[Y_k(x) - Y_{n-1}(x) \right]$ (2.)

converges absolutely and uniformly to a function $\Phi(x)$ on the interval $|x-x_0| \leq h$

If we examine the k-th partial sum of the series (2), we get $\sum_{n=1}^{k} [y_n(x) - y_{n-1}(x)]$

$$= [y_1(x) - y_0(x)] + \dots + [y_k(x) - y_{k-1}(x)]$$

$$= y_k(x) - y_0$$

Thus, the statement that the series (2) converges absolutely and uniformly to $\Phi(x)$ on $|x-x_0| \leq h$ is equivalent to the statement that the series $y_n(x) - y_0$ converges absolutely and uniformly to $\Phi(x)$ on the interval $|x-x_0| \leq h$.