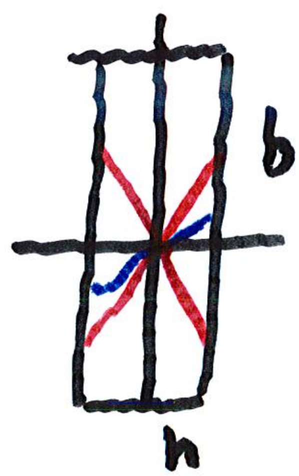
Recall that we defined

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

where $M = \sup|f(x,y)|$ in the rectangle $T = \left\{(x,y) : \begin{array}{l} |x - x_0| \le h \\ |y - y_0| \le b. \end{array}\right\}$

and $Mh \le b$.



This gives us a sequence of curves in the 2 triangular regions.

We also assume that f satisfies

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

in the rectangle T. We set

$y_0(x) = y_0$. This means

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt.$$

Hence $|y_1(x) - y_0(x)| \leq M|x - x_0|$

By iteration, we showed that

$$|y_n(x) - y_{n-1}(x)| = \left| \int_{x_0}^{x} f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)\, dt \right|$$

$$\le \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))|\, dt$$

$$\le K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|\, dt$$

Assume by induction that

$$|y_{n-1}(x) - y_{n-2}(x)| \le \frac{MK^{n-2}|x-x_0|^{n-1}}{(n-1)!}$$

Then the red formula becomes

$$|y_n(x) - y_{n-1}(x)| \le \frac{MK^{n-1}\int_{x_0}^{x}(t-x_0)^{n-1}}{(n-1)!}$$

$$\le \frac{MK^{n-1}|x-x_0|^n}{n!}$$

By induction, this last formula is true for all $n = 1, 2, 3, \ldots$

We now compare two infinite series

$$\sum_{n=1}^{\infty} | y_n(x) - y_{n-1}(x) |$$

and

$$\sum_{n=1}^{\infty} \frac{M K^{n-1} h^n}{n!}$$

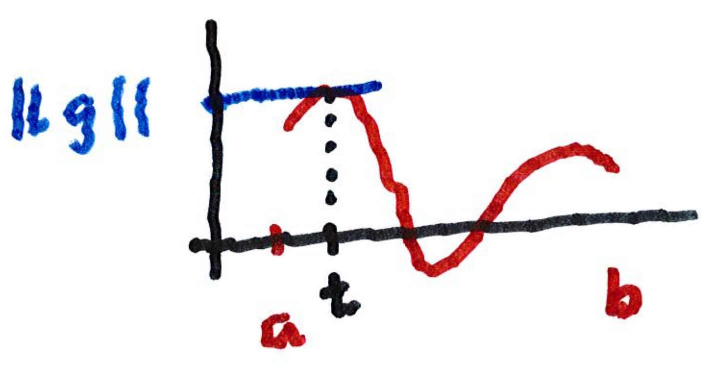
Each term on the left is dominated by the corresponding term in the series on the right.

Before stating the Weierstrass M Test, we define

If $g(t)$ is a bounded function on a set E, then

$$\|g\| = \sup \{ |g(t)| : t \in E \}$$

Intuitively $\|g\|$ = Max height of $|g(t)|$ on E



The sup-norm or uniform norm

Properties of sup-norm.

1. Triangle Inequality

$$\|f+g\| \le \|f\| + \|g\|$$

2. Homogeity

$$\|cf\| = |c|\,\|f\|$$

3. Positivity

$\|f\| \ge 0$, $\|f\| = 0$ only if $f(t) = 0$, all t.

Thm. Weierstrass M-Test

Let $u_1(x), u_2(x), \ldots$ be a sequence of functions on a set E and let $M_1, M_2, \ldots$ be constants such that

1. $|u_k(x)| \leq M_k$, for all k and all $x \in E$

2. $\sum_{k=1}^{n} M_k < \infty$

Then the series $\sum_{k=1}^{\infty} u_k(x)$

converges uniformly and

absolutely on E.

Proof Let $S_n(x) = \sum_{k=1}^{n} u_k(x)$

and $T_n(x) = \sum_{k=1}^{n} M_k$.

If $n > m$, we have

$$|S_n(x) - S_m(x)| = \left|\sum_{k=m+1}^{n} u_k(x)\right|$$

$$\leq \sum_{k=m+1}^{n} M_k = |T_n - T_m|$$

for all $x \in E$

Hence $\|S_n - S_m\| \leq | T_n - T_m |$

Since $\{T_n\}$ is a Cauchy Sequence

$$\lim \|S_n - S_m\| = 0.$$

Thus $\{S_n\}$ converges uniformly on E.

Thus, for every $\varepsilon > 0$, there is an integer N so that

$$\|S_n - S_m\| < \varepsilon.$$

Also, there is a function $S(x)$

defined on E such that

S_n converges uniformly to S on E

i.e. for any $\varepsilon > 0$, there is

$N > 0$, so that

$$|S_n(x) - S(x)| < \varepsilon,$$

i.e. $\|S_n - S\| < \varepsilon$.

Thm Suppose that $f_n \to f$ uniformly on an interval I. Suppose also that each function f_n is continuous at some x_0 in E. Then f is continuous at x_0

Pf. Let $\varepsilon > 0$ be given. We can choose an integer n such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$.

Since f is continuous at x_0, there is a neighborhood U of x_0 such that

$$\text{if } x \in U, \text{ then } |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}.$$

By the Triangle Inequality,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)|$$
$$|f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence, by the Weierstrass

M-Test, the series

$$\sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] \qquad (1)$$

above on the right converges

absolutely and uniformly on

the interval $|x-x_0| \leq h$

If we consider the k-th

partial sum of the series in (1)

we see that

$$\sum_{n=1}^{k} [y_n(x) - y_{n-1}(x)] =$$

$$y_k(x) - \dots + y_1(x) + y_0 - y_0$$

$$= y_k(x).$$

Thus the statement that the series in (1) converges absolutely and uniformly is equivalent to the statement that the sequence $y_n(x)$

converges uniformly on the

interval $|x - x_0| \le h$.

If we define $\Phi(x) = \lim_{n \to \infty} y_n(x)$

and recall that each function

$y_n(x)$ is continuous, and

$$\Phi(x) = \lim_{n \to \infty} y_n(x) \}$$

$$= y_0 + \lim_{n \to \infty} \int_{x_0}^{x} f(t, y_{n-1}(t)) \, dt$$

Observe that the final term

is $\lim_{n\to\infty} \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$.

If we could switch the order,

we would get

$$\int_{x_0}^{x} \lim_{n\to\infty} f(t, y_{n-1}(t))\,dt,$$

we would get $y_{n-1}(t) \to \Phi(t)$,

i.e. $\int_{x_0}^{x} f(t, \Phi(t))\,dt.$

Is this true?

Thm. Suppose $f_n(x)$, $n=1,2,\ldots$

and also $f(x)$ are continuous

functions on an interval $[a,b]$.

Suppose also that

f_n converges uniformly on $[a,b]$.

Then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

Proof:

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$= \left| \int_a^b [f_n(x) - f(x)]\, dx \right|$$

$$\leq \int_a^b | f_n(x) - f(x) |\, dx$$

$$\leq \int_a^b \| f_n - f \|\, dx$$

$$= \| f_n - f \| (b-a) \to 0$$

Therefore the first term $\to 0$.

Back to our equation:

Note that $\lim_{n\to\infty} y_{n-1}(t) = \Phi(t)$,

Hence $\left| \int_{x_0}^{x} \left[f(t, \Phi(t)) - f(t, y_{n-1}(t)) \right] dt \right|$

$$\leq \int_{x_0}^{x} \left| f(t, \Phi(t)) - f(t, y_{n-1}(t)) \right| dt$$

$$\leq K \int_{x_0}^{x} \left| \Phi(t) - y_{n-1}(t) \right| dt$$

$$\leq K \, \| \Phi - y_{n-1} \| \, (x - x_0),$$

which approaches 0 as $n \to \infty$.

$$\therefore \ \Phi(x) = y_0 + \lim_{n \to \infty} \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

$$= y_0 + \int_{x_0}^{x} f(t, \Phi(t))\,dt$$

$\therefore$ $\Phi(x)$ is a solution for

$|x - x_0| \leq h$