Thm. If f is continuous on $[a,b]$, then f is Darboux-integrable.

Pf. Since f is continuous on $[a,b]$, it is uniformly continuous. Thus, for every $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that if u and v are in $[a, b]$ and satisfy

$|u-v| < \delta_\varepsilon$, then

$$|f(u) - f(v)| < \frac{\varepsilon}{(b-a)}.$$

Let $P = \{I_i\}_{i=1}^{n}$ be a

partition such that $\|P\| < \delta_\varepsilon$.

Let $u_i \in [x_{i-1}, x_i]$ be a

point where f attains

its minimum value on I_i,

and let $v_i \in I_i$ be a point

where f attains maximum

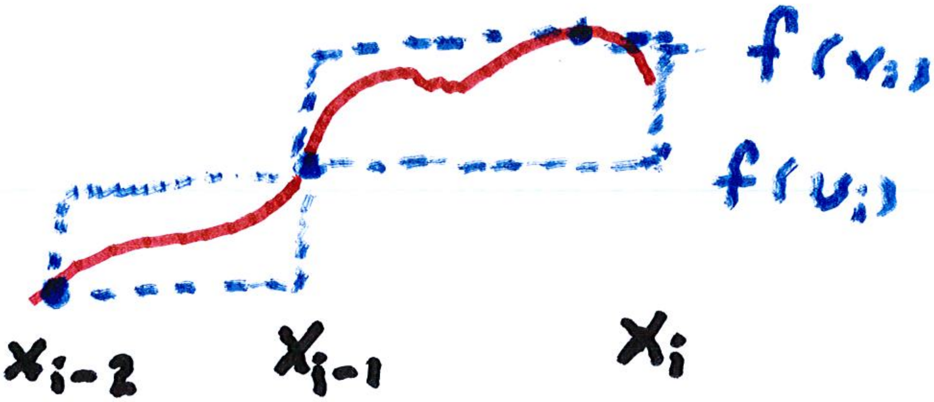
value on I_i.

Note that for all $x \in I_i$,

$$f(u_i) \leq f(x) \leq f(v_i) .$$

Let α_ε be the step function defined by $\alpha_\varepsilon(x) = f(u_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1, 2, \ldots, n-1)$ and $\alpha_\varepsilon(x) = f(u_n)$ for $x \in [x_{n-1}, x_n]$.

Let ω_ε be defined similarly using the points v_i instead of the u_i.



If we do this for every subinterval I_i, then

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x), \quad \text{for all } x \text{ in } [a,b]$$

It is clear that

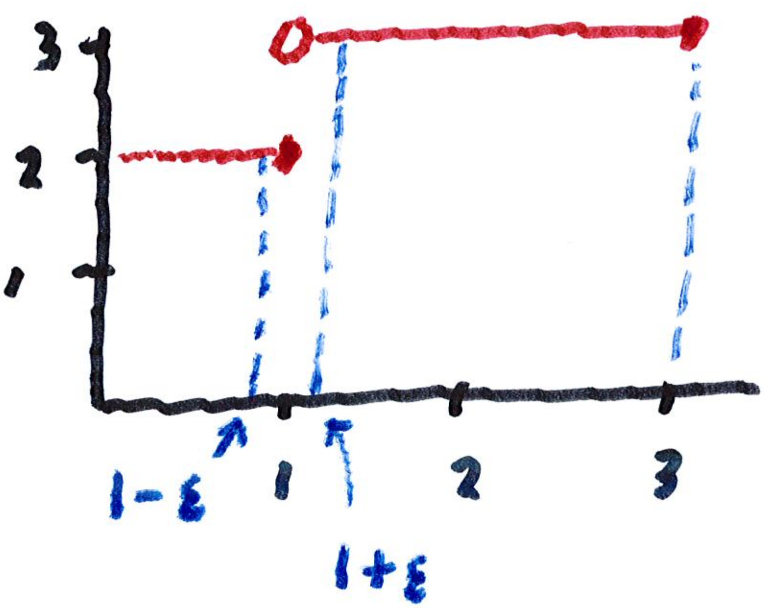
$$0 \leq \int_a^b (\omega_\varepsilon - \alpha_\varepsilon)$$

$$= \sum_{i=1}^{n} \left(f(v_i) - f(u_i) \right) (x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a} \right) (x_i - x_{i-1}) = \varepsilon$$

Therefore the Squeeze Thm. for the Darboux integral implies that f is Darboux - integrable on $[a, b]$

Ex. Calculate the Darboux integral of $f(x) = \begin{cases} 2 & \text{if } 0 \le x \le 2 \\ 3 & \text{if } 2 < x \le 3 \end{cases}$



We use the partition

$$\mathcal{P} = \{0,\ 1-\varepsilon,\ 1+\varepsilon,\ 3\}$$

$M_1 = 2 \qquad M_2 = 3 \qquad M_3 = 3$

$m_1 = 2 \qquad m_2 = 2 \qquad m_3 = 3$

$$U(f, P) = 2(1-\varepsilon) + 3(2\varepsilon) + 3(2-\varepsilon)$$

$$= 8 - 2\varepsilon + 6\varepsilon - 3\varepsilon$$

$$= 8 + \varepsilon$$

$$L(f, P) = 2(1-\varepsilon) + 2(2\varepsilon) + 3(2-\varepsilon)$$

$$= 8 - 2\varepsilon + 4\varepsilon - 3\varepsilon$$

$$= 8 - \varepsilon.$$

$U(f: P) - L(f; P)$

$= (8+\varepsilon) - (8-\varepsilon) = 2\varepsilon.$

$\therefore \int f = \lim_{\varepsilon \to 0} (8+\varepsilon) = 8.$

Location of Roots Thm p. 137

Let $I = [a, b]$ and let

$f: I \to \mathbb{R}$ be a continuous

function such that

$f(a) < 0$ $f(b) > 0$.

Then there is a $c \in (a, b)$

such that $f(c) = 0$

Pf. Set $a_1 = a$ and $b_1 = b$.

Let $p_1 = \left(\frac{a_1 + b_1}{2} \right)$. If $f(p_1) = 0$,

the by setting $c = p_1$,

we are done. Otherwise there

are 2 cases. If $f(p_1) < 0$,

then set $a_2 = p_1$ and $b_2 = b_1$

Or, if $f(p_1) > 0$, then set

$a_2 = a_1$ and set $b_2 = p_1$.

We continue this bisection process. Suppose that the intervals $I_1, I_2, \ldots, I_k$ have been obtained by bisection. Then we have $f(a_k) < 0$ and $f(b_k) > 0$. We set $p_k = \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, then we're done

Otherwise there are 2 cases

If $f(p_k) < 0$, then set

$a_{k+1} = p_k$ and $b_{k+1} = b_k$.

If $f(p_k) > 0$, then set

$a_{k+1} = a_k$ and $b_{k+1} = p_k$

we set $I_{k+1} = [a_{k+1}, b_{k+1}]$

As the process continues,

we get 2 sequences (a_n) and (b_n) for all $n \in \mathbb{N}$. Also we have

$$f(a_n) < 0 \qquad f(b_n) > 0.$$

Also, since the intervals are obtained by bisection, we get $b_n - a_n = \dfrac{(b-a)}{2^{n-1}}$, and

$$I_1 \supset I_2 \supset \dots \supset I_n \supset I_{n+1} \dots$$

The Nested Interval Property states that there is a $c \in I_n$ for all $n = 1, 2, \ldots$ This implies that

$$a_n \leq c \leq b_n.$$

Clearly (a_n) is increasing and is bounded above by b_1. so the sequence (a_n) converges to α.

Moreover, $b_n = a_n + \frac{(b-a)}{2^{n-1}}$

also converges to $\alpha + 0 = \alpha$.

Thus the constant sequence

c satisfies

$a_n \le c \le b_n$, and

(a_n) and (b_n) both converge

to α, so it follows that

$\lim_{n\to\infty} c = \alpha$. i.e., $c = \alpha$.

Thus we have

$$\lim_{n\to\infty} a_n = c, \quad \lim_{n\to\infty} b_n = c.$$

Since f is continuous at c,

we get

$$\lim f(a_n) = f(c), \quad \lim f(b_n) = f(c)$$

Since $f(a_n) < 0$, we have

$f(c) \leq 0$. and since $f(b_n) > 0$,

we have $f(c) \geq 0$.

It follows that $f(c) = 0$