Review Problems for Math 341

1a. Given a set S of real numbers, define $\sup S = u$

b. Show that for any $\varepsilon > 0$, there is a number $x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon \leq u$

page 37-38.

2. Define the Nested Interval Property. p. 48

3. Suppose that (x_n) converges to x and (y_n) converges to y. Show that $(x_n y_n)$ converges to xy.

p. 61, 62.

4. Show that if (x_n) is an increasing sequence and that $x_n \leq M$ for all n, then there is a number $L \leq M$ such that $\lim_{n \to \infty} x_n = L$

p. 71, 72

5. Suppose $\sum_{n=1}^{\infty} x_n$ is a series with $x_n \geq 0$ such that

$$\sum_{n=1}^{N} x_n \leq M \text{ for all } N = 1, 2, \ldots.$$

Show there is an $L \leq M$ such that $\sum_{n=1}^{\infty} x_n$ converges to L. p. 98

6. Use Newton's Method one time with an initial guess of 2 to find the approximate value of $\sqrt{3}$ p. 194

7. Define the Thomae function by setting $f(x) = \frac{1}{q}$, when $x = p/q$ in lowest terms, and by $f(x) = 0$, when x is irrational. Show that f is continuous at x if x is irrational and f is discontinuous at x if x is rational. p. 127, 128

8. Suppose that f is an increasing bounded function on (a, b). Show there is an L such that $\lim_{x \to b^-} f(x) = L$.

p. 117, 118

9. Show that $S(x) = \sqrt{x}$ is Lipschitz on the interval $[a, \infty)$, where $a > 0$.

p. 143, 144

10. Show that if f is Lipschitz on any interval, then f is uniformly continuous.

p. 143 - 144.

11. Find all functions that satisfy $|f(x) - f(y)| \leq |x-y|^2$

p. 162

12. Suppose that f is differentiable at x_0 and that $f(x_0) \neq 0$. Calculate $\left(\frac{1}{f}\right)'$ at x_0.

p. 115, 164

13. Let n be a positive integer and let $b > 0$. Use L'Hopital's Rule to show that

$$\lim_{x \to \infty} \left(\frac{x^n}{b^x} \right) \leq M, \text{ i.e.}$$

$|x^n| \leq b^x$, if x is sufficiently large.

p. 187

14. Use $\ln x$ and L'Hopital's Rule to show that $\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$

Set $x = \frac{1}{n}$.

p. 183

15. Use the Taylor polynomial of order 3 to calculate $\ln(1+x)$ to within an error of $\frac{1}{50}$

p. 190.

16. Show that on every interval $[-d, d]$, the Taylor Series of $\cos x$ converges to $\cos x$.

p. 190

17. Define $g(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Is g continuous at 0? Is g differentiable at 0? Is $g \in R[0, 1]$. Is g uniformly continuous on $[0, 1]$?

p. 130, 162, 212

18. Suppose f is differentiable at all $x \in (a, b)$, If $f'(x) > 0$ in (a, b), show f is increasing.

p. 174

19. When f is a bounded function on $[a,b]$ and when $\dot{P}$ is a partition of $[a,b]$ that is tagged, what is $S(f;\dot{P})$?

p. 200

20. What is the definition of the statement $f \in R[a,b]$?

p. 201

21. State the Integrability Criterion for a function to be Darboux-integrable.

22. Let $g(x) = \begin{cases} 3 & \text{if } 0 \leq x < 2 \\ 1 & \text{if } 2 \leq x \leq 4 \end{cases}$

p. 229

Use the criterion above with a partition P having just 4 points to show g is Darboux integrable. p. 229, p. 228

23. Use the above criterion to show that any continuous function f on $[a,b]$ is Darboux integrable. p. 229, 228

24. To solve the differential equation $y'(x) = f(x, y(x))$ with $y(x_0) = y_0$, we defined a sequence of curves $y_n(x)$.

How are curves $y_n(x)$ defined? (class notes) online