Sets can be arbitrarily large: For any set S, let $\mathcal{P}(S)$ be the set of all subsets of S.

Cantor's Thm:

There does NOT exist a map $\varphi : S \longrightarrow \mathcal{P}(S)$ that is onto.

Proof. Suppose

$$\phi : S \longrightarrow \mathcal{P}(S)$$

is a surjection.

Note $\phi(x)$ is a subset of S. Either x belongs to $\phi(x)$ or it does not belong to $\phi(x)$. We let

$$D := \left\{ x \in S : x \notin \phi(x) \right\}$$

Since φ is a surjection,

there exists $x_0 \in S$
such that $\varphi(x_0) = D$.

There are 2 cases:

1. Suppose $x_0 \in D$.

   Then $x_0 \in \varphi(x_0)$.

   By definition of D,

   $x_0 \notin D$. Contradiction

2. Suppose $x_0 \notin D$.

Then $x_0 \notin \varphi(x_0)$.

By definition of D,

$x_0 \in D$. Contradiction.

Ex. Suppose $S = \{a, b, c\}$

$\wp(S) = \{ \phi, \{a\}, \{b\}, \{c\},$

$\{a,b\}, \{a,c\}, \{b,c\}$

and $\{a,b,c\} \}$

S has 3 elements,

$\mathcal{P}(S)$ has 8 elements.

There does not exist a surjection from S onto $\mathcal{P}(S)$.

2.1 Algebraic and Order Properties of $\mathbb{R}$.

On $\mathbb{R}$, there are two operations; addition $+$ multiplication $\cdot$. They satisfy:

(A1) $a+b = b+a$, (commutative addition)

(A2) $(a+b)+c = a+(b+c)$ (associative addition)

(A3) There is an element 0 in $\mathbb{R}$ so $a+0 = a$ (0-element exists)

(A4) For each a in $\mathbb{R}$, there is an element $-a$ in $\mathbb{R}$ so that

$$a + (-a) = 0 \text{ and } (-a) + a = 0$$

(negative element)

(M1) $a \cdot b = b \cdot a$ (commutative multiplication)

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(associative multiplication)

(M3) There is an element 1 in $\mathbb{R}$

so that $a \cdot 1 = 1 \cdot a = a$

(unit element exists)

(M4). For each $a \neq 0$ in $\mathbb{R}$,

there exists an element

$1/a$ such that

$a \cdot \left(\frac{1}{a}\right) = 1$ and

$\left(1/a\right) \cdot a = 1$

{existence of recipricál}

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

and

$(b+c) \cdot a = (b \cdot a) + (c \cdot a)$

(distributive property)

In a word, $\mathbb{R}$ is a <u>field</u>

By applying some of the above properties, one can show that the

(1) zero element 0, the

(2) unit element 1, and

(3) the reciprical $\frac{1}{a}$ are

all unique.

For example, suppose $a \neq 0$

and $a \cdot b = 1$. Then

$$b \underset{(M_3)}{=} 1 \cdot b \underset{(M4)}{=} \left(\left(\frac{1}{a}\right) \cdot a \right) \cdot b$$

$$\underset{(M2)}{=} \left(\frac{1}{a}\right) \cdot (a \cdot b) = \left(\frac{1}{a}\right) \cdot 1 \underset{(M3)}{=} \frac{1}{a}.$$

This proves (3)

Also, if $a \in \mathbb{R}$, then $a \cdot 0 = 0$

In fact,

$$\underbrace{a + a \cdot 0 = a \cdot 1}_{\text{by } (M_3)} + \underbrace{a \cdot 0 = a \cdot (1+0)}_{\text{by } (D)}$$

$$\underbrace{= a \cdot 1}_{\text{by } (A_3)} \underbrace{= a}_{\text{by } (M_3)}$$

Adding $(-a)$ to both sides, we get

$a \cdot 0 = 0.$

Also, $0 = (-1)(-1+1) = (-1)(-1) + (-1).$

Adding 1 to both sides, we get

$(-1)(-1) = 1$

We define subtraction by

$$a - b = a + (-b)$$

and also we write

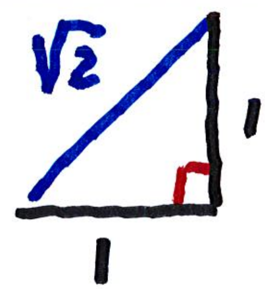
$$ab = a \cdot b,$$

and $a^2 = aa$ and

$a^3 = a^2 a$ and

$a^{n+1} = a^n a$, etc.

Thm. There does not exist a rational number r such that $r^2 = 2$



Suppose by contradiction that $r = p/q$. Then

$$r^2 = (p/q)^2 = 2 \to p^2 = 2q^2.$$

We can assume that p and q have no common factor. Then at most one of p and q is even.

Since $p^2 = 2q^2$, we see that p^2 is even. This implies that p is also even (because if $p = 2n+1$ is odd, then $p^2 = 4n^2 + 4n + 1$ is also odd.)

Hence we can write $p = 2m$,

so that

$$p^2 = 4m^2 = 2q^2.$$

Dividing by 2,

$$2m^2 = q^2.$$

Hence q^2 must be even,

which implies q is even.

This shows that both

p and q are even, which

is a <u>contradiction</u>.

It follows that

$\mathbb{R}$ must include numbers

that are irrational

(i.e., not rational).

For this purpose we need to

study Order Properties.

i.e., $<$ and $>$.

Order Properties of $\mathbb{R}$

There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers such that

(i) If $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$

(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$

(iii) If $a \in \mathbb{P}$, then exactly one of the following holds:

$a \in \mathbb{P}$, $a = 0$, $(-a) \in \mathbb{P}$

Trichotomy Property

If $-a \in \mathbb{P}$, we say a is negative, and we write $a < 0$ or $0 > a$.

(i) If $a \in \mathbb{P}$, we write $a > 0$ or $0 < a$

(ii) If $a \in \mathbb{P} \cup \{0\}$, we write $a \geq 0$.

(iii) If $-a \in \mathbb{P} \cup \{0\}$, then we write $a \leq 0$.

If (i)–(iii) hold, then we say $\mathbb{R}$ is an ordered field

Applying the Trichotomy Property to $a-b$, we get

If $a-b \in \mathbb{P}$, then $a > b$.

If $-(a-b) \in \mathbb{P}$, then $(b-a) \in \mathbb{P}$

$\Rightarrow b > a$

If $a-b = 0$, then $a = b$

Here are the Rules for Inequalities:

Thm. Let $a, b, c \in \mathbb{R}$.
2.1.7

(a) If $a > b$ and $b > c$, then

$a > c$

(b) If $a > b$, then $a + c > b + c$

(c) If $a > b$ and $c > 0$, then

$ca > cb$

If $a > b$ and $c < 0$, then

$ac < ab$

Proof of (a): $a - b > 0$, $b - c > 0$

then $(a-b) + (b-c) > 0$

or $a - c > 0 \rightarrow a > c$

(b) If $a-b>0$, then

$$(a+c)-(b+c) = a-b > 0$$

$$\to a+c > b+c$$

(c) If $a>b$ and $c>0$, then

$$ca-cb = c(a-b) > 0.$$

$$\to ca > cb$$

If $c<0$, then $-c>0$. Hence

$$c(b-a) = -c(a-b) > 0$$

$$\to cb-ca>0 \to cb > ca.$$

The Order Properties in 2.1.5 and 2.1.6 lead to 2.1.10 and 2.1.11, which are useful for solving inequalities:

1. Suppose that $ab > 0$. If $a > 0$, then $b > 0$.
2. If $ab > 0$ and $a < 0$, then $b > 0$
3. If $ab < 0$ and $a > 0$, then $b < 0$
4. If $ab < 0$ and $a < 0$, then $b > 0$

Ex. Find all real numbers x such that $3x + 4 \le 12$.

Justify each step.

$$3x + 4 \le 12 \Leftrightarrow 3x \le 8 \Leftrightarrow x \le \frac{8}{3}$$

By (b) of 2.1.7

By (c) of 2.1.7

Ex. Solve $x^2 - 4x - 5 < 0$.

$$x^2 - 4x - 5 = (x-5)(x+1) < 0$$

$\Leftrightarrow$ By Property (3) above

$\underbrace{\text{If } x-5 > 0 \text{, then } x+1 < 0}_{\text{No solution.}}$

By Property (4)

Or, if $x-5<0$, then $x+1>0$

$\therefore$ Solution is $-1 < x < 5$

Finally, we have

Thm. 2.1.8

(i) if $a \in \mathbb{R}$ and $a \neq 0$, then

$a^2 > 0$

(ii) if $n \in \mathbb{N}$, then $n > 0$.