If $-a \in \mathbb{P}$, we say a is negative, and we write $a < 0$ or $0 > a$.

(i) If $a \in \mathbb{P}$, we write $a > 0$ or $0 < a$

(ii) If $a \in \mathbb{P} \cup \{0\}$, we write $a \geq 0$.

(iii) If $-a \in \mathbb{P} \cup \{0\}$, then we write $a \leq 0$.

If (i)-(iii) hold, then we say $\mathbb{R}$ is an ordered field.

Applying the Trichotomy Property to $a-b$, we get

If $a-b \in \mathbb{P}$, then $a > b$.

If $-(a-b) \in \mathbb{P}$, then $(b-a) \in \mathbb{P}$

$\Rightarrow b > a$

If $a-b = 0$, then $a = b$

Here are the Rules for Inequalities:

Thm. Let $a, b, c \in \mathbb{R}$.
2.1.7

(a) If $a > b$ and $b > c$, then $a > c$

(b) If $a > b$, then $a + c > b + c$

(c) If $a > b$ and $c > 0$, then $ca > cb$

If $a > b$ and $c < 0$, then $ac < ab$

Proof of (a): $a - b > 0$, $b - c > 0$

then $(a-b) + (b-c) > 0$

or $a - c > 0 \rightarrow a > c$

(b) If $a-b>0$, then

$$(a+c)-(b+c) = a-b > 0$$

$$\rightarrow a+c > b+c$$

(c) If $a>b$ and $c>0$, then

$$ca-cb = c(a-b) > 0.$$

$$\rightarrow ca > cb$$

If $c<0$, then $-c>0$. Hence

$$c(b-a) = -c(a-b) > 0$$

$$\rightarrow cb-ca>0 \rightarrow cb > ca.$$

The Order Properties in 2.1.5 and 2.1.6 lead to 2.1.10 and 2.1.11, which are useful for solving inequalities:

1. Suppose that $ab > 0$. If $a > 0$, then $b > 0$.
2. If $ab > 0$ and $a < 0$, then $b > 0$
3. If $ab < 0$ and $a > 0$, then $b < 0$
4. If $ab < 0$ and $a < 0$, then $b > 0$

Ex. Find all real numbers x such that $3x+4 \le 12$. Justify each step.

$$3x+4 \le 12 \Leftrightarrow 3x \le 8 \Leftrightarrow x \le \frac{8}{3}$$

By (b) of 2.1.7 By (c) of 2.1.7

Ex. Solve $x^2-4x-5 < 0$.

$$x^2-4x-5 = (x-5)(x+1) < 0$$

$\Leftrightarrow$ If $x-5>0$, then $x+1<0$

By Property (3) above

No solution.

By Property (4)

Or, if $x-5<0$, then $x+1>0$

$\therefore$ Solution is $-1 < x < 5$

Finally, we have

Thm. 2.1.8

(i) if $a \in \mathbb{R}$ and $a \neq 0$, then

$a^2 > 0$

(ii) if $n \in \mathbb{N}$, then $n > 0$.

Absolute Value 2.2.

We can define $|a|$ as follows:

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

We'll need these identities:

(a) $|-a| = |a|$

(b) $|ab| = |a||b|$

(c) $|a|^2 = a^2$

(d) $-|a| \leq a \leq |a|$

(e) if $b < 0$, then $|b| = -b$.

Proof.

(a) Suppose $a \geq 0$. Then $-a \leq 0$

$\rightarrow |-a| = -(-a) = a = |a|$

If $a < 0$, then $-a > 0$, so

$|-a| = -a = |a|$

by def. of $|a|$
when $a < 0$

(b) If either a or $b = 0$, then

both sides equal 0.

Now suppose $a, b > 0$.

$|ab| = ab = |a||b|$

since $ab > 0$

Now suppose $a > 0$, $b < 0$.

$|ab| = -ab = a(-b) = |a||b|$

When $a < 0$ and $b > 0$, and

$a, b < 0$, the argument is

similar.

(c) Since $a^2 \geq 0$,

$a^2 = |a^2| = |a||a| = |a|^2$.

(d). When $a \geq 0$, $a = |a|$

$\therefore -|a| \leq 0 \leq a \leq |a|$

Similarly, when $a \leq 0$,

$|a| = -a$. or $-|a| = a \leq 0 \leq |$.

$-|a| = a \leq 0 \leq |a|$

Hence, $-|a| \leq a \leq |a|$

The following inequality is very useful.

Triangle Inequality.

If $a, b \in \mathbb{R}$, then

$$|a+b| \leq |a| + |b|.$$

Pf. Suppose first that $a+b \geq 0$

$\rightarrow$ $|a+b| = a+b \leq |a| + |b|$

using (d)

Now suppose that $a+b<0$

$$\to |a+b| = -(a+b)$$

$$= -a-b \le |a|+|b|$$

using (d).

which implies the Triangle Inequality. We can prove

$$|a-b| \le |a|+|b| \qquad (1)$$

by replacing b by $-b$.

We will also need:

$$\big|\,|a| - |b|\,\big| \leq |a-b| . \qquad (*)$$

Pf.

$$a = (a-b) + b$$

$$|a| \leq |a-b| + |b|$$

$$\rightarrow \quad \big(|a| - |b|\big) \leq |a-b| \qquad (2)$$

Similarly $\quad b = b-a + a$

$$|b| \leq |b-a| + |a|$$

$$|b| - |a| \le |b-a|$$

$$-\big(|a| - |b|\big) \le |a-b| \qquad (3)$$

By combining (2) and (3),

we obtain

$$\big||a| - |b|\big| \le |a-b|,$$

which proves (*).

Another version is the

Backwards Triangle Property

$$|a-b| \geq |a| - |b|.$$

Pf.

$$|a| = |(a-b)+b|$$

$$\leq |a-b| + |b|$$

$$\Rightarrow |a-b| \geq |a| - |b|$$

One more identity

Suppose $c \geq 0$. Then

(1) $|a| \leq c$ if and only if

$$-c \leq a \leq c.$$

Proof:

Case 1: Assume $a \geq 0$.

$$|a| \leq c \rightarrow a \leq c$$

$$\rightarrow -c \leq 0 \leq a$$

Case 2: Assume $a < 0$.

$-a = |a| \leq c$

$\rightarrow -c \leq a < 0 \leq c$

Thus, in both cases, we get the desired inequality.

Now, let's prove the "if" direction. We know

$a \leq c$. Also, $-c \leq a$

or $-a \leq c$

We obtain $|a| \le c$

Thus, we've proved both directions.

Ex. Find the set A of all x such that $|3x+4| < 2$

$\therefore$ Left half is

Set $c = 2$ and $a = 3x+4$.

$|a| < c \rightarrow -c < a < c$

or $-2 < 3x+4 < 2$

$\therefore \quad -6 < 3x < -2$

$\rightarrow \quad -2 < x < -\frac{2}{3}$.

Ex. Set $f(x) = \dfrac{2x^2 - 4x + 3}{5x - 2}$.

when $1 \le x \le 2$

For the numerator;

$$|2x^2 - 4x + 3| \le |2x^2| + |4x| + 3$$

$$\le 8 + 8 + 3 = 19$$

For the denominator:

$$|5x - 2| \geq |5x| - |2|$$
$$\geq 5 - 2 = 3$$

Hence,

$$|f(x)| \leq \frac{19}{3}$$

Def'n. Let $a \in \mathbb{R}$ and $\varepsilon > 0$.

Then the ε-neighborhood of

a is the set

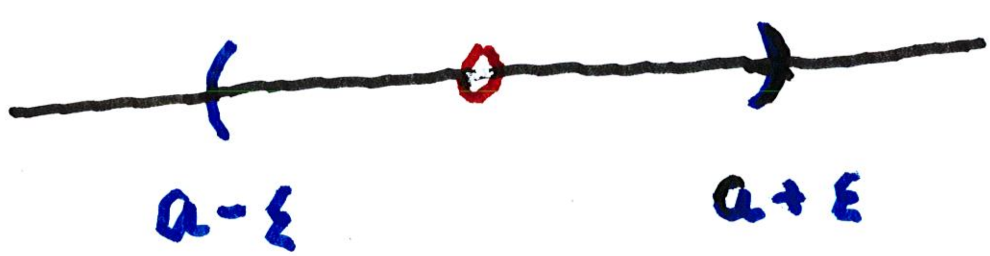
$$V_\varepsilon(a) = \left\{ x \in \mathbb{R} : |x - a| < \varepsilon \right\}.$$

If we replace a in (1) by $x-a$ and c by ε, it follows that $x \in V_\varepsilon(a)$ if only if

$$-\varepsilon < x-a < \varepsilon$$

or $$a-\varepsilon < x < a+\varepsilon$$

On the real line this is



Thm. Let $a \in \mathbb{R}$. If x belongs to $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Pf. Suppose $x \neq a$. If we set $\varepsilon = \frac{|x-a|}{2}$ in the definition of $V_\varepsilon(a)$, then

$$|x-a| < \frac{|x-a|}{2}.$$

Dividing by $|x-a|$, we have $1 < \frac{1}{2}$. This contradiction $\rightarrow$ $x = a$.

2.3 The Least Upper Bound Property for $\mathbb{R}$.

Consider the following systems of numbers:

$$\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$$

Each system is modified to fill in a certain gap.

The definition of $\mathbb{R}$ is the most complicated.

If we define define

numbers x and y in $\mathbb{R}$

as infinite decimal expansions

such as

$x = \pm A.a_1 a_2 a_3 \ldots$ and

$y = \pm B.b_1 b_2 b_3 \ldots$, then the

nine axioms for a field

and three order properties

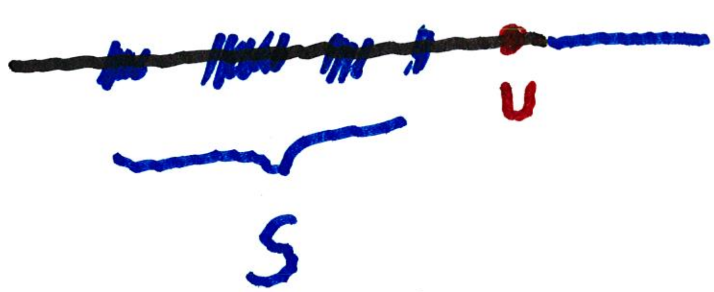
are all satisfied.

One can show that $\mathbb{R}$ satisfies the Least Upper Bound Property:

Definition. Let S be a nonempty subset of $\mathbb{R}$.

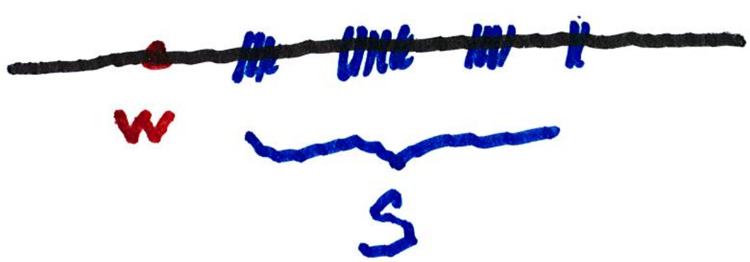
(a) S is bounded above if there is a number $u \in \mathbb{R}$ such that

$s \leq u$ for all $s \in S$.



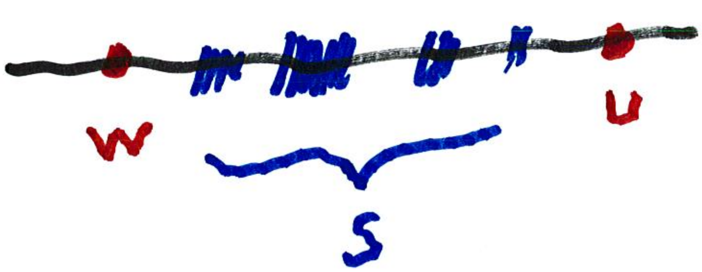
u is an upper bound of S

(b) S is bounded below if there is a number $w \in \mathbb{R}$ such that $s \geq w$ for all $s \in S$.



w is a lower bound of S.

(c) S is bounded if it is bounded above and below



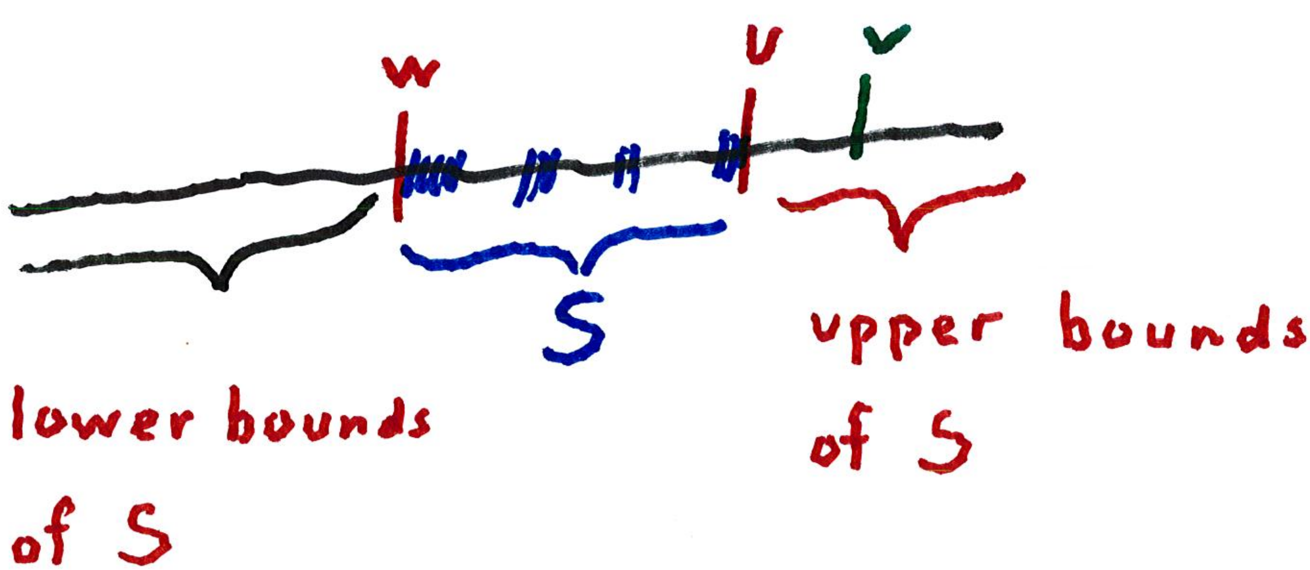
If S is not bounded, then S is unbounded.

Suppose that S is nonempty.

(a) A number u is a **least upper bound of S** if

(1) u is an upper bound of S

and (2) If v is any upper bound of S then $v \geq u$



(b) A number w is a greatest lower bound of S if

(1′) w is a lower bound of S

and (2′) if t is any lower bound of S, then $t \le w$.

If a least upper bound of S exists, we write l.u.b. S = supremum S
= sup S

If a greatest lower bound of S exists, we write g.l.b. S = infimum S
= inf S

The main fact about $\mathbb{R}$ is that if S is a subset of $\mathbb{R}$ that is bounded above, then there is a number u in $\mathbb{R}$ such that $u = \sup S$

Similarly, if S is bounded below, then there is a $w \in \mathbb{R}$ such that $w = \inf S$