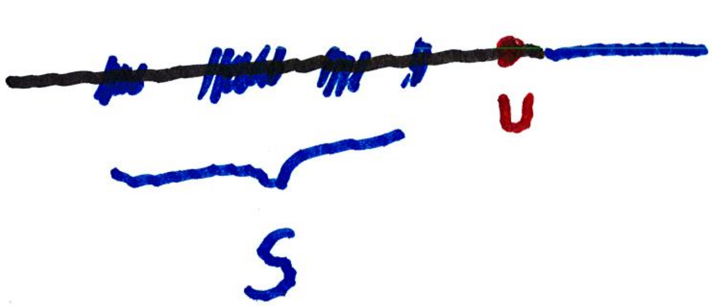
One can show that $\mathbb{R}$ satisfies the Least Upper Bound Property:

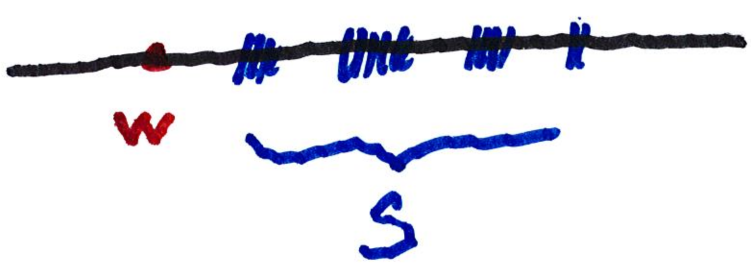
Definition. Let S be a nonempty subset of $\mathbb{R}$.

(a) S is bounded above if there is a number $U \in \mathbb{R}$ such that $s \le U$ for all $s \in S$.



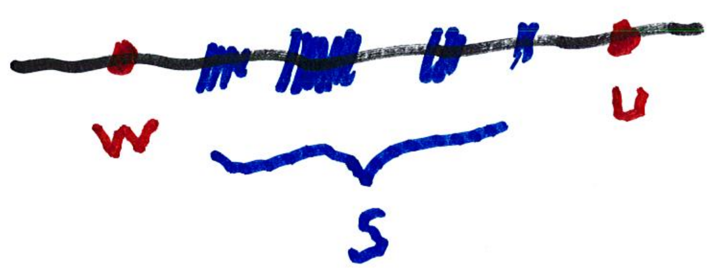
U is an upper bound of S

(b) S is bounded below if there is a number $w \in \mathbb{R}$ such that $s \geq w$ for all $s \in S$.



w is a lower bound of S.

(c) S is bounded if it is bounded above and below



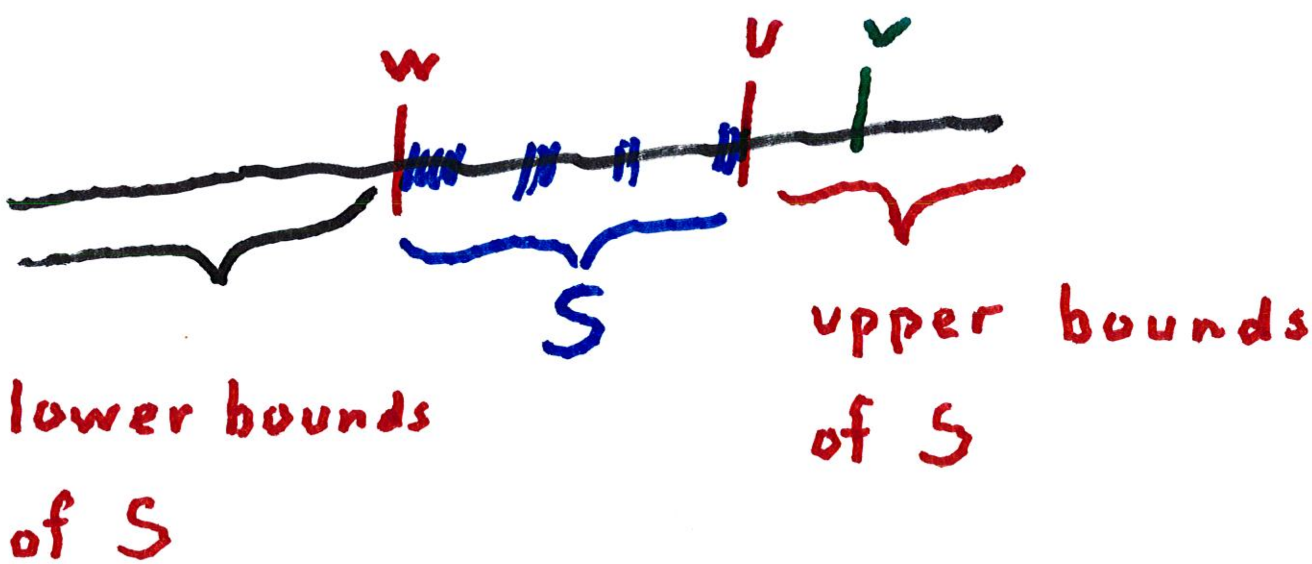
If S is not bounded, then S is unbounded.

Suppose that S is nonempty.

(a) A number u is a least upper bound of S if

(1) u is an upper bound of S

and (2) If v is any upper bound of S then $v \geq u$



(b) A number w is a greatest lower bound of S if

(1′) w is a lower bound of S

and (2′) if t is any lower bound of S,

then $t \le w$.

If a least upper bound of S exists,

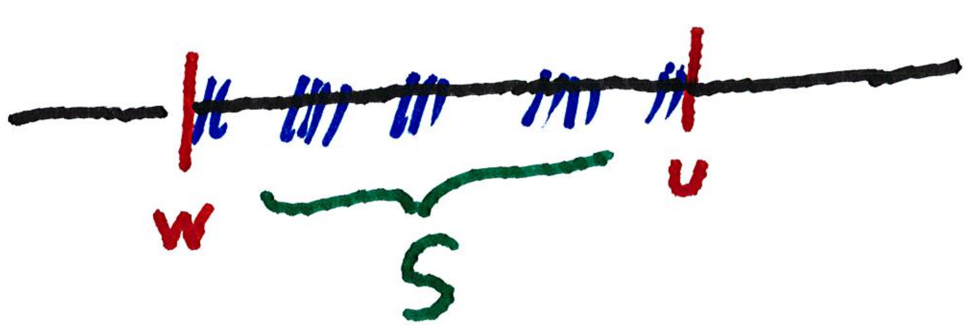
we write l.u.b. S = supremum S
= sup S

If a greatest lower bound of S exists,

we write g.l.b. S = infimum S
= inf S

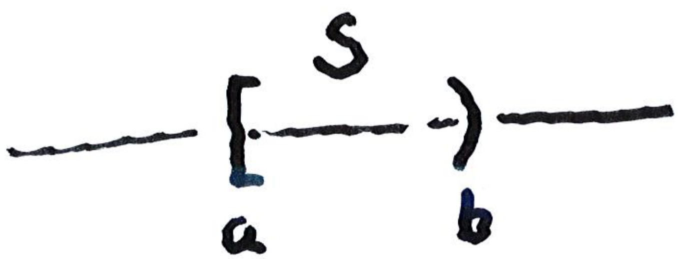
The main fact about $\mathbb{R}$ is that if S is a subset of $\mathbb{R}$ that is bounded above, then there is a number u in $\mathbb{R}$ such that $u = \sup S$

Similarly, if S is bounded below, then there is a $w \in \mathbb{R}$ such that $w = \inf S$



Note that inf S and sup S do not always belong to S.

Ex. Let $S = [a, b)$, where $a < b$.



Find inf S and sup S

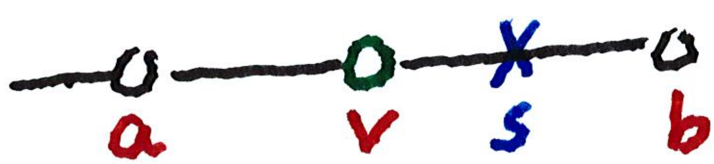
$x \in [a, b) \rightarrow a \leq x < b.$ (*)

$\therefore$ b is an upper bound of $[a, b)$

Let v be any upper bound of S.

Suppose that $v \le a$. Then $s = \frac{a+b}{2}$ is in S, but $v < s$, which would show v is <u>not</u> an upper bound of S.

Suppose that $a < v < b$. Then $s = \frac{v+b}{2}$ is in S, but $v < s$, which shows that



v is <u>not</u> an upper bound of S.

It follows that $v \geq b$, which shows that b is the least upper bound of S

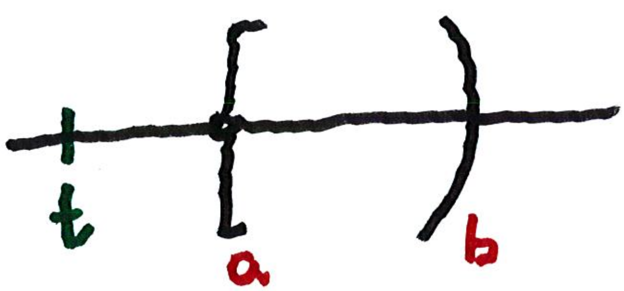
Now note that the first inequality in $(*)$ shows that a is a lower bound of S.

$\therefore$ $(1')$ holds

Since $s = a$ is in S, it follows that any lower bound t of S must satisfy $t \leq a$. Hence,

$a =$ g.l.b of S.

or $\inf S = a$



Different Definitions.

Suppose u is an upper bound of a set S:

(1) If v is any upper bound, then $u \le v$

(2) If $z < u$, then z is not an upper bound.

(3) If $z < u$, then there exists $s_z \in S$ such that $z < s_z$

(4) If $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

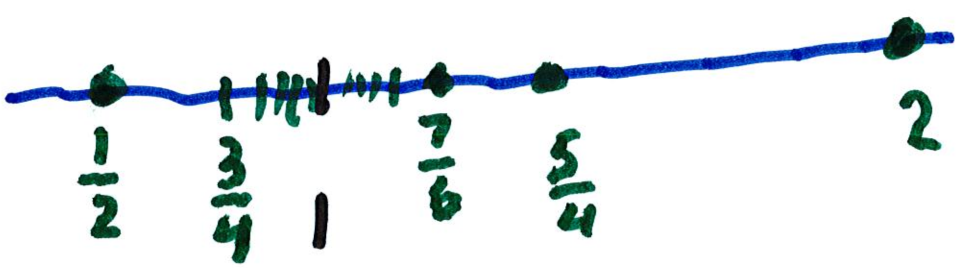
#4, p. 39.

Let $S = \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$

If n is even, then the sequence is $1 - \frac{1}{n}$, $n = 2, 4, 6, \ldots$

If n is odd, the sequence is

$1 + \frac{1}{n}$, $n = 1, 3, 5, \ldots$



The sequence is contained in

$\left[\frac{1}{2}, 2\right]$ where $\frac{1}{2}$ and 2 are included.

u satisfies $u \geq s$, $s \in S$

which means $\underline{u = 2}$

v is an upper bound if $v \geq s$,

i.e. $v \geq 2$

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w satisfies $w \leq s$, which means

$\underline{w \leq \frac{1}{2}}$. t is a lower bound if $t \leq s$

i.e., $t \leq \frac{1}{2}$

Since $u = 2$ and $v \geq 2$,

it follows that $u \leq v$,

so supremum $= 2$

Similarly, $w = \frac{1}{2}$ and $t \leq \frac{1}{2}$.

$\frac{1}{2}$ is an infimum if $t \leq w$.

Thus $\sup S = 2$ and

$\inf S = \frac{1}{2}$