**Def'n.** Recall $u$ is an upper bound if $u \ge s$, for all $s \in S$

Also, $w$ is a lower bound if $w \le s$, for all $s \in S$.

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**Def'n.** $u$ is a supremum of $S$ (or a least upper bound) if

(1) $u$ is an upper bound, and

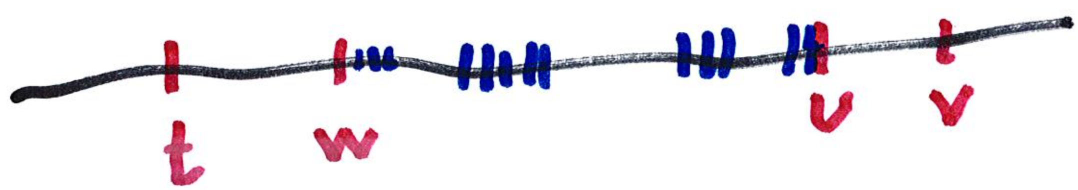
(2) if $v$ is any upper bound, then $v \ge u$.

Similarly, $w$ is an infimum of $S$ (or a greatest lower bound) if

(1') $w$ is a lower bound, and

(2') if $t$ is a lower bound, then

$t \leq w$.



It's easy to show that there can only be one supremum:

Suppose there are 2 suprema $U_1$ and $U_2$. Suppose $U_1 < U_2$.

The fact that $U_2$ is a supremum and $U_1$ is an upper bound implies that $U_2 \geq U_1$

Similarly, one shows that $U_1 \geq U_2$.

Given a set $S$ with an upper bound $u$, there are 4 ways to express the statement that $u$ is a supremum

(1) If $v$ is any upper bound of $S$, then $v \geq u$.

(2) If $z < u$, then $z$ is NOT an upper bound. (For, if $z$ were an upper bound,

this would contradict (1),

i.e., it would imply $z \geq U$.)

(3) If $z < U$, then there is

an element $s_z \in S$ such that

$s_z > z$.

(For if all $s \in S$ satisfy

$s \leq z$, that would imply

$z$ is an upper bound,

contradicting (2))

(4) If $\varepsilon > 0$, then there is an element $s_\varepsilon$ such that $s_\varepsilon \in S$ and $s_\varepsilon > U - \varepsilon$.

$$\left( \begin{array}{c} (3) \text{ holds for} \\ z = U - \varepsilon \end{array} \right)$$

Finally, we must show $(4) \to (1)$.

Let $V < U$ and let $V = U - \varepsilon$.

Then by (4), there exists $s_\varepsilon \in S$ so that $U - \varepsilon < s_\varepsilon$

$\Rightarrow V < s_\varepsilon$ $\therefore$ $V$ is NOT an upper bound

Thus, if $v$ is an upper bound, then it must be that

$v \geq u$.

$\mathbb{R}$ satisfies:

Completeness Property.

Every nonempty set of real numbers that has an upper bound has a supremum in $\mathbb{R}$.

# Archimedean Property.

1. If $x > 0$, then there exists $n_x \in \mathbb{N}$ so that $x < n_x$.

Pf. Suppose this is NOT true. Then for every $n \in \mathbb{N}$, we would have $n \leq x$, for all $n$ in $\mathbb{N}$. By the Completeness Property, $\mathbb{N}$ has a supremum $u$.

Then $u-1$ is not an upper bound of $\mathbb{N}$, so there is an integer $m \in \mathbb{N}$ with $u-1 < m$. Adding 1, we get $u < m+1$. This contradicts the fact that $n \leq x$ for all $n$. Hence, there is an integer $n_x$ with $n_x > x$.

2. For any $\varepsilon > 0$, there is an integer $K$ in $N$ so that $\frac{1}{n} < \varepsilon$, for all $n \geq K$.

Pf. Set $x = \frac{1}{\varepsilon}$. We showed above that there is an integer $n_x$, such that

$n_x > x$. If we set $K = n_x$, then if $n \geq K$, then

$$n \geq n_x > x = \frac{1}{\varepsilon}.$$

3. If $y > 0$, then there exists $n_y \in N$ such that

$$n_y - 1 \leq y \leq n_y \qquad (*)$$

Pf. The Archimedean Property implies that the subset $E_y = \{m \in N : y < m\}$ is nonempty. The Well-Ordering Property implies that $E_y$ has a least element, we denote by $n_y$. Then

$n_y - 1$ does not belong to $E_y$

Hence we have

$$n_y - 1 \leq y < n_y$$

Density Theorem.

If $x$ and $y$ are any real numbers with $x < y$, then there is a rational number $r \in \mathbb{Q}$ such that $x < r < y$

Pf. We can assume that $x > 0$. ( Let $m \in \mathbb{N}$ satisfy $m + x > 0$. Then replace $x$ with $x + m$ and $y$ with $y + m$ )

Since $y-x>0$, it follows from 2. that there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y-x$.

which gives $nx+1 < ny$.

If we apply (*) to $nx$, we obtain $m \in \mathbb{N}$ with

$$m-1 \le nx < m.$$

Therefore,

$$m \le nx + 1 < ny,$$

which leads to

$$nx < m < ny.$$

Thus the rational number

$r = m/n$ satisfies

$$x < r < y$$