Thm. A convergent sequence of real numbers is bounded.

Pf. Suppose that $\lim x_n = x$ and let $\varepsilon = 1$. Then there is a $K \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq K$. The Triangle Inequality with $n \geq K$ implies that

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x|$$
$$< 1 + |x|.$$

If we set

$$M = \max\left\{ |x_1|, |x_2|, \ldots |x_{K-1}|, 1 + |x| \right\}$$

then it follows that

$$|x_n| \leq M, \qquad \text{for all } n \in \mathbb{N}.$$

We want to learn how taking limits interacts with the operations of addition, subtraction, multiplication and division.

Given two sequences $X = (x_n)$ and $Y = (y_n)$, we define

$$X + Y = (x_n + y_n)$$

$$X - Y = (x_n - y_n)$$

$$XY = (x_n y_n)$$

$$cX = (cx_n)$$

and

$$X/Y = \left(\frac{x_n}{y_n}\right) \quad \left(\begin{matrix}\text{providing} \\ y_n \neq 0\end{matrix}\right)$$

Suppose $X = (x_n)$ and $Y = (y_n)$

converge to x and y

respectively. Let $\varepsilon > 0$.

Addition.

Choose K_1 and K_2 so that

$|x_n - x| < \frac{\varepsilon}{2}$ if $n \geq K_1$ and

$|y_n - y| < \frac{\varepsilon}{2}$ if $n \geq K_2$.

Now set $K = \mathrm{Max}\{K_1, K_2\}$

If $n \geq K$, then $n \geq K_1$ and $n \geq K_2$. Hence,

$$|(x_n + y_n) - (x + y)|$$

$$= |(x_n - x) + (y_n - y)|$$

$$\leq |x_n - x| + |y_n - y|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim (x_n + y_n) = x + y$.

For subtraction, we use the same argument. Just replace $x_n + y_n$ by $x_n - y_n$ and $x + y$ by $x - y$.

Multiplication. This is a bit more complicated. Note that

$$|x_n y_n - xy| = \left| \begin{aligned} &(x_n y_n - x_n y) \\ &+ (x_n y - xy) \end{aligned} \right|$$

$$\leq |x_n(y_n - y)| + |(x_n - x)y|$$

$$\leq |x_n||y_n - y| + |x_n - x||y|$$

By the boundedness theorem, there is $M_1 > 0$ such that

$$|x_n| \leq M_1 \quad , \quad \text{all } n.$$

Now set $M = \text{Max}\{M_1, |y|\}$.

We conclude that

$$|x_n y_n - xy| \leq M|y_n - y| + M|x_n - x|$$

Now let $\varepsilon > 0$ be given.

Then there exists K_1

such that

$$|x_n - x| < \frac{\varepsilon}{2M} \quad \text{if } n \geq K_1.$$

Similarly, there exists K_2

such that

$$|y_n - y| < \frac{\varepsilon}{2M} \quad \text{if } n \geq K_2.$$

Now set $K = \text{Max}\{K_1, K_2\}$

If $n \geq K$, then

$$|x_n y_n - xy|$$

$$\leq M|y_n - y| + M|x_n - x|$$

$$< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

This proves

$$\lim (x_n y_n) = xy.$$

3.2 Limit Thms.

Given 2 sequences

$X = (x_n)$ and $Y = (y_n)$ such that

$\lim (x_n) = x$ and $\lim (y_n) = y$,

we proved that

1. $\lim (x_n + y_n) = x + y$

2. $\lim (x_n y_n) = xy$.

3 To prove $\lim (cx_n) = cx$,

let $Y = (y_n) = c$, for all c.

Then $\lim cx_n = \lim y_n x^n$

$= \lim y_n \cdot \lim x_n$

$= cx \quad \therefore \lim cx_n = cx.$

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4. Now suppose $Z = (z_n)$ and that $\lim (z_n) = z \neq 0$.

Choose $K_1 \in \mathbb{N}$ so that if $n \geq K_1$, then $|z_n - z| < \frac{|z|}{2}$.

It follows that

$$|z_n| = |(z_n - z) + z|$$

$$= |z + (z_n - z)|$$

$$\geq |z| - |z_n - z|$$

$$\geq |z| - \frac{|z|}{2} = \frac{|z|}{2}.$$

We use this to estimate the limit of $\frac{1}{z}$:

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| = \left| \frac{z - z_n}{z_n z} \right|$$

$$\le |z - z_n| \cdot \frac{2}{|z|^2}$$

since $\dfrac{1}{|z_n|} \le \dfrac{2}{|z|}$ when

$n \geq K_1$. Now choose $\varepsilon > 0$

and choose K_2 so that

$|z_n - z| < \dfrac{|z|^2}{2} \varepsilon$ when $n \geq K_2$.

Now set $K = \mathrm{Max}\{K_1, K_2\}$.

If $n \geq K$, then

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| \le |z - z_n| \cdot \frac{2}{|z|^2}$$

$$< \frac{|z|^2}{2} \cdot \varepsilon \cdot \frac{2}{|z|^2} = \varepsilon$$

This shows that $\lim\left\{\frac{1}{z_n}\right\} = \frac{1}{z}$.

Ex. Use the Limit Laws to compute $\lim \dfrac{n^2 + 2n}{3n^2+1}$

$$\frac{n^2 + 2n}{3n^2+1} = \frac{n^2\left(1 + \frac{2}{n}\right)}{n^2\left(3 + \frac{1}{n^2}\right)}$$

$$= \frac{1 + \frac{2}{n}}{3 + \frac{1}{n^2}} .$$

Since $\lim \frac{1}{n} = 0$,

we have $\lim \frac{2}{n} = 0$ and $\lim \frac{1}{n^2} = 0$

$\therefore \lim \left(1 + \frac{2}{n}\right) = 1 + 0 = 1$

and $\lim \left(3 + \frac{1}{n^2}\right)$.

Hence the Quotient Rule

$$\rightarrow \lim \frac{1 + \frac{2}{n}}{3 + \frac{1}{n^2}} = \frac{1}{3}$$

Ex. Compute $\lim \frac{\sqrt{n}}{2n+3}$

Factor out highest power

$$= \frac{\sqrt{n} \cdot 1}{n\left(2+\frac{3}{n}\right)} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\left(2+\frac{3}{n}\right)}$$

Note $\lim \frac{1}{\sqrt{n}} = 0$ and

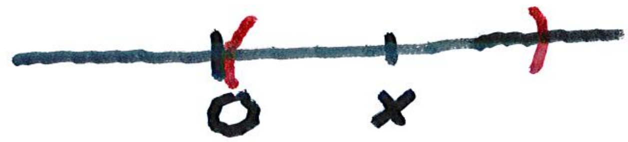
$$\lim \frac{1}{2+\frac{3}{n}} = \frac{1}{2}.$$

$\therefore$ Product Rule implies

$$\lim \frac{\sqrt{n}}{2n+3} = 0 \cdot \frac{1}{2} = 0$$

Thm. Suppose $\lim x_n = x$ and that $x_n \leq 0$. Then $x \leq 0$.

Pf. Suppose statement is not true, i.e., suppose $x > 0$.



Pick $\varepsilon = x$

Then there is K, so if $n \geq K$, then $|x_n - x| < x$

Hence $-\varepsilon < x_n - x < \varepsilon$.

$-x < x_n - x \rightarrow 0 < x_n$.

This contradicts hypothesis that $x_n \leq 0$

Corollary. Suppose (x_n) and (y_n) are both convergent and that $x_n \leq y_n$, all n. Then $\underline{x \leq y}$

Pf. Set $z_n = x_n - y_n$.

Then $z_n \le 0$, for all n.

Hence the theorem implies

$\lim z_n = z$, i.e., $z \le 0$.

$\therefore$ $x - y \le 0$.

i.e. $\lim(x_n) \le \lim(y_n)$.

Suppose $a \le x_n \le b$ and that (x_n) is convergent.

Then $a \le \lim(x_n) \le b$.

Pf. To prove $\lim(x_n) \leq b$,

set $(y_n) = (b)$ for all n.

The hypothesis that $x_n \leq b$

(using previous result)

implies that $\lim(x_n) \leq \lim(y_n)$,

or $\lim(x_n) \leq b$.

Similarly, if we set $y_n = a$,

for all n, then the

hypothesis $\Rightarrow$ $y_n \leq x_n$, which implies $a \leq \lim(x_n)$.

Squeeze Thm.

Suppose that $X = (x_n)$,

$Y = (y_n)$, and $Z = (z_n)$ are

sequences with

$$x_n \leq y_n \leq z_n.$$

Suppose also that $\lim(x_n) = \lim(z_n)$

Then $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

Proof: Let $w = \lim(x_n)$
$= \lim(z_n)$.

For any $\varepsilon > 0$, choose K so that if $n \geq K$, then

$|x_n - w| < \varepsilon$ and $|z_n - w| < \varepsilon$.

$\rightarrow -\varepsilon < x_n - w \leq y_n - w \leq z_n - w < \varepsilon$

$\rightarrow \quad -\varepsilon < y_n - w < \varepsilon$

$\rightarrow \quad \lim y_n = w$

Ex. Compute $\lim \frac{(-1)^n}{n}$.

$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}.$$

Also, we know that

$\lim \frac{1}{n} = 0$ and $\lim \frac{-1}{n}$.

$\therefore$ Squeeze Thm. $\to \lim \frac{(-1)^n}{n} = 0$

Ratio Test for Sequences:

Let (x_n) be a sequence of positive numbers such that

$L = \lim \left\{ x_{n+1} / x_n \right\}$ exists.

If $L < 1$, then $\lim (x_n) = 0$.

Let r be a number satisfying

$L < r < 1$ and let $\varepsilon = r - L$.

There is a number K so that

if $n \geq K$, then

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon.$$

It follows that

$$\frac{x_{n+1}}{x_n} - L < \varepsilon = r - L$$

$$\therefore \quad \frac{x_{n+1}}{x_n} < r \qquad \text{for all } n \geq K.$$

Hence $0 < x_{n+1} < r x_n$ for all $n \geq K$.

Then $x_{K+1} < r x_K$

$$x_{K+2} < r x_{K+1} < r^2 x_K$$

$$x_{K+3} < r^3 x_K$$

$$\vdots$$

$$x_{K+n} < r^n$$

Since $\lim r^n = 0$, it follows

that $\lim x_{K+n} = 0$.