Basic Facts:

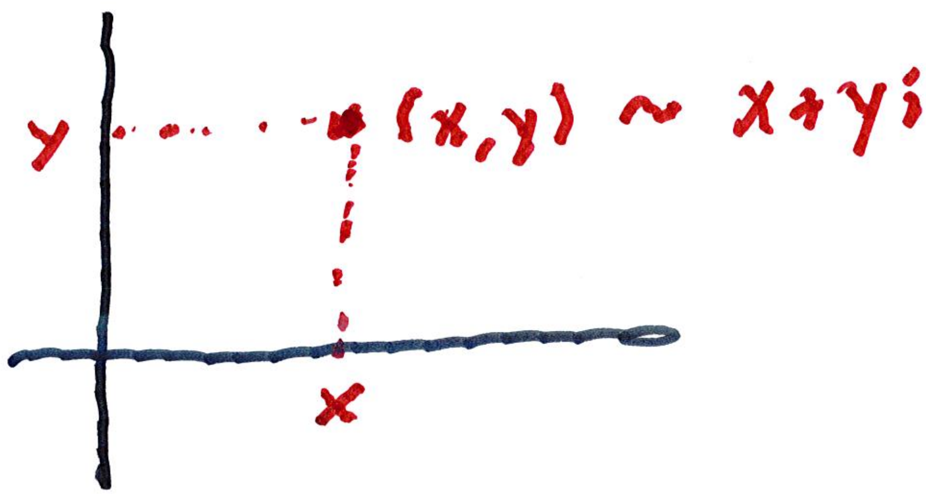
A complex number has the form

$$z = x + yi, \quad \text{where } x, y \in \mathbb{R} \text{ and } i^2 = -1$$

We define

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

We can view $x+yi$ as equivalent to the point (x,y) in $\mathbb{R}^2$.



If $z_1 = x_1 + iy_1$

and $z_2 = x_2 + iy_2$,

then we define

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$z_1 z_2 = (x_1 y_1 - x_2 y_2) + i(x_1 y_2 + x_2 y_1)$$

The formula for $z_1 z_2$

comes by using $i^2 = -1$

$$(x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1 x_2 + i^2 y_1 y_2)$$

$$+ i(x_2 y_1 + x_1 y_2)$$

One can verify that the operations of Addition and Multiplication are:

Commutative $z_1 + z_2 = z_2 + z_1$

$z_1 z_2 = z_2 z_1$

Associative $z_1 + (z_2 + z_3)$

$= (z_1 + z_2) + z_3$

and also $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Distributive

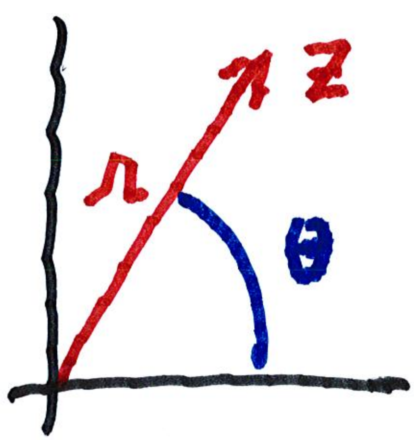
$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

The geometric interpretation of addition is the Parallelogram Law.

For multiplication, we first note that if $z \neq 0$,

we can write z in Polar Form

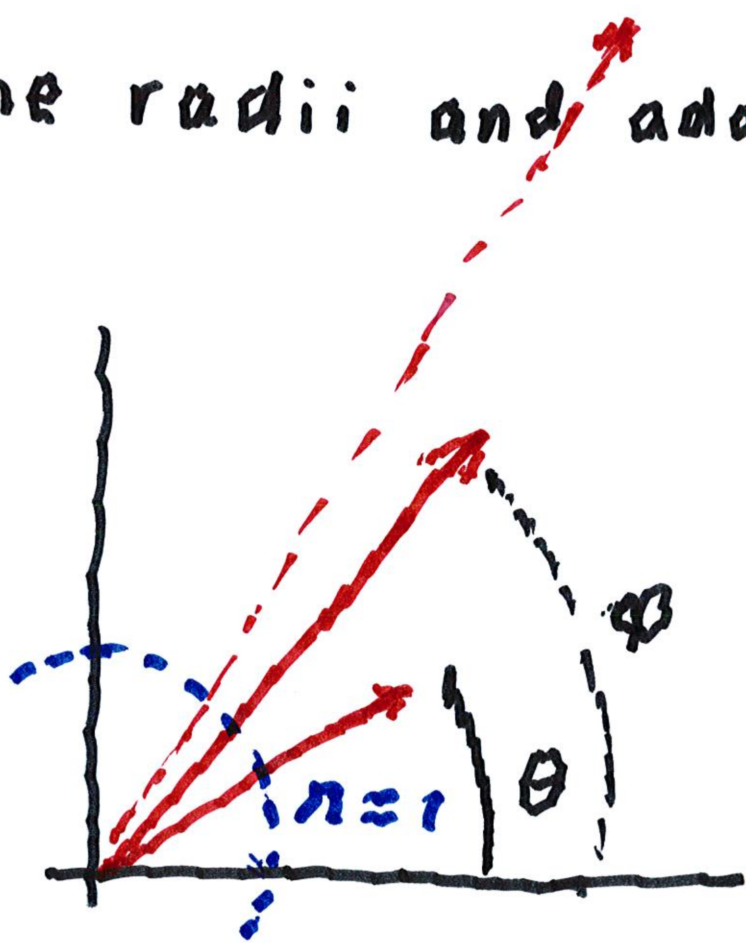
$z = re^{i\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$ (or $-\pi \leq \theta < \pi$) etc.



If $w = se^{i\varphi}$, then

$$zw = re^{i\theta} se^{i\varphi} = rse^{i(\theta+\varphi)}$$

Thus we can multiply the radii and add the angles.



Absolute Value.

We define

$$|z| = |x+iy| = (x^2+y^2)^{1/2}.$$

Thus, $|z|$ is = to the Euclidean distance of (x,y) from $0=(0,0)$.

One can prove the

Triangle Inequality

$$|z+w| \leq |z| + |w|$$

for all z, w in $\mathbb{C}$.

Also, the

Backwards Triangle Inequality

$$|z-w| \geq |z| - |w|$$

In fact

$$|z| = |(z-w)+w|$$

$$\leq |z-w| + |w|$$

$$\Rightarrow \quad |z-w| \geq |z| - |w|$$

Note also that

$$\big|\, |z| - |w| \,\big| \leq |z-w|$$

(i.e., $|z|$ is continuous)

We define the complex

conjugate $\bar{z}$ by $\bar{z} = x - iy$

(if $z = x + yi$)

$|z| = |x + iy| = x - iy$

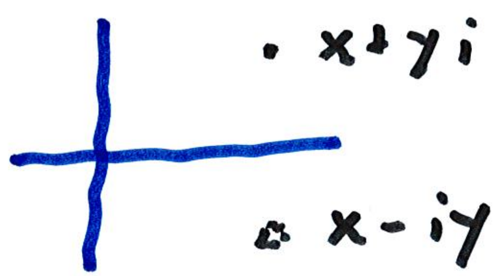
It satisfies

$\overline{z+w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\,\bar{w}$.

One can see that

$\bar{z} = z$ if and only if

z is real.



It has many applications.

Suppose $p(z) = 0$,

where $p(z) = a_n z^n + \dots + a_0$,

where each coefficient

a_k is real, and $a_n \neq 0$.

Then $\cancel{p(\bar{z}) = p(z) =}$

$p(\bar{z}) = 0$.

$$\overline{p(z)} = \bar{0} = 0$$

$$= \overline{\sum_{k=0}^{n} a_k z^k}$$

$$= \sum_{k=0}^{n} \overline{a_k z^k}$$

$$= \sum_{k=0}^{} \overline{a_k}\ \overline{z^k}$$

$$= \sum_{k=0}^{n} a_k (\bar{z})^k$$

$\therefore p(\bar{z}) = 0$

Thus all roots are real

or they come in conjugate

pairs.

Basic Terminology

1. If $z_0 \in \mathbb{C}$ and $r > 0$,

then $D_r(z_0) =$

$$\{ z ; \ |z - z_0| < r \}$$

2. $\bar{D}_r(z_0) = \{z;\ |z - z_0| \leq r\}$

3. If $\Omega \subset \mathbb{C}$ and $z_0 \in \Omega$, then z_0 is an <u>interior point</u> of Ω, if for some $r > 0$, $D_r(z_0) \subset \Omega$

4. A set Ω is <u>open</u> if every point of Ω is an interior point of Ω.

5. A set Ω is <u>closed</u>

if its complement is

open

6. A point $z \in \mathbb{C}$ is

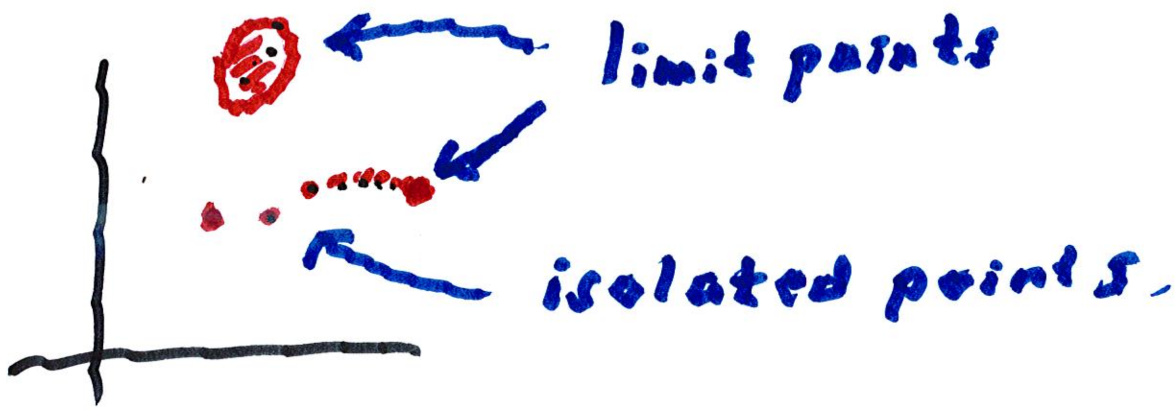
a <u>limit</u> <u>point</u> of Ω if

there is a sequence

$z_n \in \Omega$ such that

$$\lim_{n \to \infty} z_n = z.$$

7. One can show that any set Ω is closed if and only if it contains all of its limit points



8. The closure of a set Ω

is the union of Ω and all

its limit points.

9. The boundary of a set Ω

is the closure minus

its interior

10. A set Ω is bounded if there is a constant $M > 0$ such that $|z| < M$ for every $z \in \Omega$.

11. If Ω is bounded, we define the <u>diameter</u> by

$$\operatorname{diam}(\Omega) = \sup_{z,w \in \Omega} |z - w|$$

12. A set $\Omega \subset \mathbb{C}$ is compact if and only every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

13. An open covering of Ω is a family of open sets U_α such that $\Omega \subset \bigcup_\alpha U_\alpha$

$\alpha \in A$

14. The following are equivalent

(i) Ω is <u>compact</u>

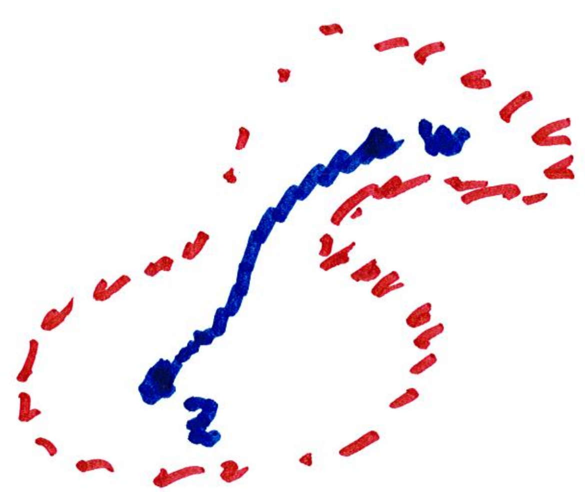
(ii) Ω is <u>closed</u> and <u>bounded</u>

(iii) Every open covering of Ω has a finite subcovering.

16. An open set Ω is connected if and only if any two points z and w can be joined by a cont. curve $\gamma(t)$, $a \le t \le b$,

$z = \gamma(a)$, $w = \gamma(b)$,

where $\gamma(t) \in \Omega$ for all t



17 2.1 Continuous Functions

Suppose f is defined

on a set Ω. If $z_0 \in \Omega$,

then f is continuous at

z_0 if for every $\varepsilon > 0$,

there is $\delta > 0$ so that

$$|f(z) - f(z_0)| < \varepsilon$$

when $|z - z_0| < \delta$ and $z \in \Omega$.

18. Equivalently, f is continuous at z_0, if for every sequence $\{z_n\} \subset \Omega$ that converges to z_0 it is true that

$$\lim_{n\to\infty} f(z_n) = f(z_0).$$

19. A function f on Ω is continuous if f is continuous at every point.

If f, g are continuous on Ω so are sums and products.

If f is continuous, so is $|f(z)|$.

20. We say f has ~~a maximum~~

a <u>maximum</u> at $z_0 \in \Omega$

if $|f(z)| \leq |f(z_0)|$, all $z \in \Omega$

and f has a <u>minimum</u> at z_0

if $|f(z_0)| \leq |f(z)|$, all $z \in \Omega$.

21. Theorem A continuous function on a compact set Ω is bounded and has a maximum and a minimum on Ω.