Definite Integrals

Using residues, we can compute integrals such as

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

We make the substitution $z = e^{i\theta}$

$$0 \leq \theta \leq 2\pi$$

Then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + z^{-1}}{2}$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

Also, $z = e^{i\theta} \rightarrow dz = ie^{i\theta} d\theta$

$\rightarrow dz = iz\, d\theta \rightarrow d\theta = \frac{1}{iz} dz$

The above integral becomes

$$\int_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}.$$

This is a holomorphic line integral,

and C is the unit circle about

the origin.

Ex. Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta}$, $|a|<1$

The integral becomes

$$\int_C \frac{1}{1+a\left(\frac{z-z^{-1}}{2i}\right)} \frac{dz}{iz}$$

$$= \int_C \frac{2i}{2i+a(z-z^{-1})} \frac{dz}{iz}$$

Divide by a, on top and bottom :

$$= \int_C \frac{\frac{2}{a}\, dz}{\frac{2zi}{a} + (z^2 - 1)}$$

$$= \int_C \frac{2/a}{z^2 + \left(\frac{2i}{a}\right) z - 1}$$

Now find roots of denominator

$$z_1 = \left(\frac{-1+\sqrt{1-a^2}}{a}\right)i$$

$$z_2 = \left(\frac{-1-\sqrt{1-a^2}}{a}\right)i$$

One verifies that $z_1 z_2 = -1$

Since $|z_2| > 1$, it follows

that $|z_1| < 1$.

If we write $f(z) = \dfrac{\phi(z)}{z - z_1}$

then $\phi(z) = \dfrac{2/a}{z - z_2}$

and hence $\operatorname{Res}_{z_1} f(z) = \dfrac{2/a}{z_1 - z_2}$

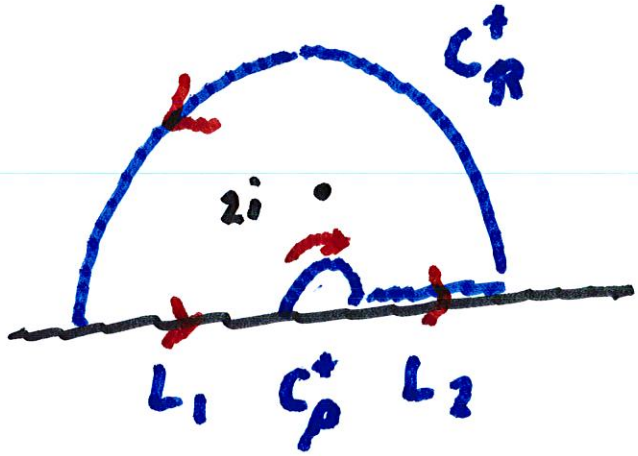
$$= \frac{1}{i\sqrt{1-a^2}}$$

It follows that

$$\int_C \frac{2/a \quad dz}{z^2 + \left(\frac{2i}{a}\right) z - 1} = \frac{2\pi i}{i\sqrt{1-a^2}}$$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

Ex. Use the path



to compute

$$\int_0^\infty \frac{\ln x \, dx}{(x^2+4)^2}$$

If we always parameterize

C_R^+ and C_ρ^+ (semicircular arcs)

positively, and if

$$f(z) = \frac{\ln z}{(z^2+4)^2}, \text{ then}$$

the above line integral

becomes

$$\int_{L_1} f(z)\,dz - \int_{C_\rho^+} f(z)\,dz + \int_{L_2} f(z)\,dz \qquad (1)$$

$$+ \int_{C_R^+} f(z)\,dz = 2\pi i \operatorname{Res}_{2i} f(z)$$

since $(z^2+4)^2 = (z-2i)^2(z+2i)^2$

If we write $f(z) = (z-2i)^{-2}\,\phi(z)$

then $\phi(z) = \dfrac{\log z}{(z+2i)^2}$

and $\operatorname{Res}_{2i} f(z) = \left(\dfrac{\log z}{(z+2i)^2}\right)'\Bigg|(2i)$

which equals

$$\phi'(2i) = \frac{\pi}{64} + \frac{i\,(1-\ln 2)}{32}$$

Recall that $\log z = \ln r + i\theta$,

$(z = re^{i\theta})$

so $$\int_{L_2} f(z)\,dz = \int_\rho^R \frac{\ln r}{(r^2+4)^2}\,dr$$

and $$\int_{L_1} f(z)\,dz = \int_\rho^R \frac{\ln r + i\pi}{(r^2+4)^2}\,dr$$

Thus (1) becomes

we get

$$2\int_\rho^R \frac{\ln x \, dx}{(x^2+4)^2} = \frac{\pi}{16}(\ln 2 - 1)$$

$$- \operatorname{Re}\int_{C_\rho} f(z)\,dz - \operatorname{Re}\int_{C_R} f(z)\,dz .$$

We still have to compute

$$\int_{C_\rho} f(z)\,dz \quad \text{and} \quad \int_{C_R} f(z)\,dz .$$

For $\int_{C_\rho}$, note that if $\rho < 1$

and $z = \rho e^{i\theta}$ is a point on C_ρ,

then

$$|\log z| = |\ln\rho + i\theta| \le -\ln\rho + \pi,$$

and $|z^2+4| \ge \left||z|^2 - 4\right| = 4 - \rho^2$

Hence $\left|\operatorname{Re}\int_{C_\rho} f(z)\,dz\right| \le \dfrac{-\ln\rho + \pi}{(4-\rho^2)^2}\pi\rho$

$$= \pi\,\frac{\pi\rho - \rho\ln\rho}{(4-\rho^2)^2} \to 0 \text{ as } \rho \to 0.$$

Similarly.

$$\left| \operatorname{Re} \int_{C_R^+} f(z)\,dz \right| \le \left| \int_{C_R^+} f(z)\,dz \right|$$

$$\le \frac{\ln R + \pi}{(R^2-4)^2}\,\pi R \to 0 \quad \text{as } R \to \infty.$$

Hence $\displaystyle\int_0^\infty \frac{\ln t \; dt}{(t^2+4)^2} \to \frac{\pi}{32}(\ln 2 - 1)$

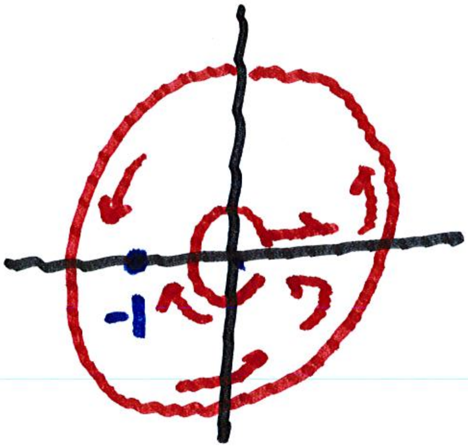
Integration along a branch cut.

Ex. Compute $\displaystyle\int_0^\infty \frac{x^{-a}}{x+1}$, $0<a<1$

First, note that

$$z^{-a} = \exp(-a \log z)$$

We define a path by



The inner circle has radius ρ

and the outer circle radius R.

We write

$$f(z) = \frac{\exp(-a \log z)}{z+1}$$

$$= \frac{\exp\{-a(\ln r + i\theta)\}}{re^{i\theta} + 1}$$

On the upper edge, $\theta = 0$, so

$(z = re^{i0}$

$$f(z) = \frac{\exp(-a\ln r + i0)}{r+1}$$

$$= \frac{r^{-a}}{r+1}.$$

On the lower edge, $\theta = 2\pi$, so

$$f(z) = \frac{\exp(-a\ln r - ia2\pi)}{r+1}$$

The function $\frac{1}{z+1}$ has a simple pole

at $z=-1$, and $f(z)= \frac{z^{-a}}{(z+1)}$

so $\mathrm{Res}_{-1} f(z) =$

z^{-a} (where $z=-1$) $\left(e^{i\pi}=-1\right)$

$= \exp(-ia\pi) = \underline{e^{-ia\pi}}$

The sum of the integrals is

$$-\int_{C_\rho} f(z)\,dz + \int_{C_R} f(z)\,dz$$

$$= 2\pi i e^{-ia\pi} + (e^{-i2a\pi} - 1)\int_\rho^R \frac{r^{-a}}{r+1}\,dr.$$

On C_ρ, we have

$$\left|\int_{C_\rho} f(z)\,dz\right| \leq \frac{\rho^{-a}}{1-\rho}\,2\pi\rho = \frac{2\pi}{1-\rho}\rho^{1-a}$$

and on C_R.

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{R^{-a}}{R-1} 2\pi R = \frac{2\pi R}{R-1} \cdot \frac{1}{R^a}.$$

Since $0 < a < 1$, both integrals

$\to 0$ as $\rho \to 0$ and $R \to \infty$

Hence $\int_0^R \frac{r^{-a}}{r+1} = \frac{1}{e^{-i2\pi a} - 1}$

$$\left[\int_{C_R} f(z)\,dz - 2\pi i e^{-ia\pi} \right]$$

By letting $R \to \infty$ in the last equation, we get

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx =$$

$$2\pi i \frac{e^{-ia\pi}}{1-e^{-2a\pi}} \cdot \frac{e^{ia\pi}}{e^{ia\pi}}$$

$$= \pi \frac{2i}{e^{ia\pi} - e^{-ia\pi}}$$

which equals $\dfrac{\pi}{\sin a\pi}$