We determined all Analytic Automorphisms of the unit disc.

Thm: If $f$ is the only automorphism of $\mathbb{C}$, then

$f(z) = az + b$, for any $b$ and $a \neq 0$.

Pf. Since $f$ is holorphic on $\mathbb{C}$, we can write $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Clearly $h(z) = f(1/z)$ is holomorphic for all $z$, with $z \neq 0$, so has an isolated singularity.

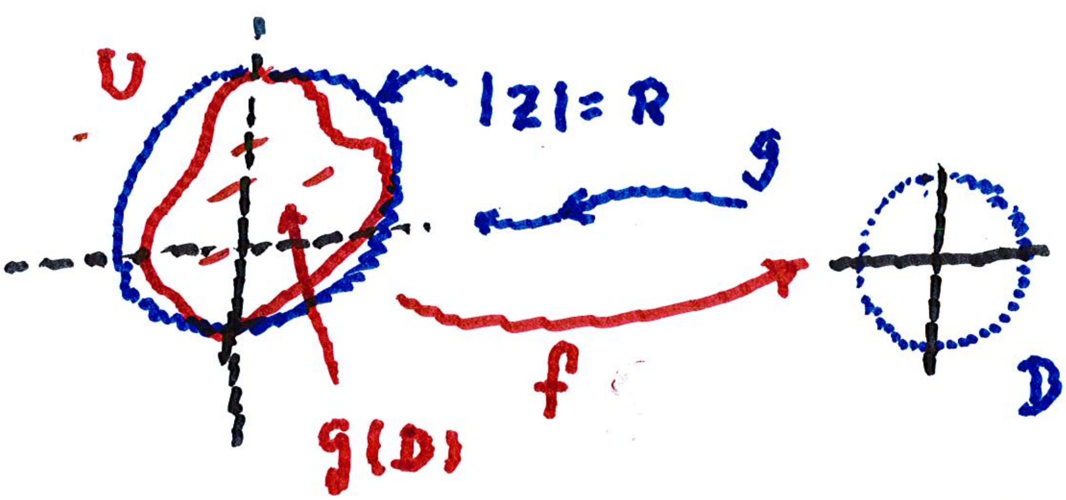
If we let $g$ be the inverse of $f$, then by the Open Mapping Theorem, $g(D)$ is open. Moreover, $\bar{D}$ is compact.

$\overline{g(D)}$ is also compact.

If we let $R = \sup\{|g(D)|\}$,

then ~~if~~ $U = \{z;\ |z| > R\}$

is open.




Now let $R_1 > R$.

Now suppose that $|z| < \frac{1}{R_1}$

Then $\left|\frac{1}{z}\right| > R_1 > R.$

$\Rightarrow \frac{1}{z} \notin g(D)$

$\Rightarrow f\left(\frac{1}{z}\right) \notin f(g(D)) = D.$

If $h(z) = f\left(\frac{1}{z}\right)$ had an

essential singularity at $0$,

then $\left\{h(z);\ |z| < \frac{1}{R_1}\right\}$

would be dense in $\mathbb{C}$.

Thus, $f(1/z)$ does not have an essential singularity.

which means that

$$f(1/z) = \sum_{j=0}^{N} a_j (1/z)^j$$

has only a finite number of terms. If $m$ = smallest $j$ such that $a_j \neq 0$ and $m \geq 2$, then we can write $\sum_{j=m}^{N} a_j z^j = p(z)$

as $\{g(z)\}^m$, for a

function $g$ which vanishes

to order 1. This would

implay that $f(z) = a_0 + (g(z))^m$

is <u>not</u> 1-to-1.

Hence $f(z) = \underline{a_0 + a_1 z}$

# Linear Fractional Transformations

Let $a, b, c, d$ be complex numbers such that $ad-bc \neq 0$. We define

$$G(z) = \frac{az+b}{cz+d}.$$

Then $G$ is analytic except at $z = -d/c$.

Let $S$ be the Gauss sphere,

i.e., $S = \mathbb{C}$ and a single point

$\infty$. We extend the

definition of $G$ to $S$

by defining

$$G(\infty) = \frac{a}{c} \quad \text{if } c \neq 0$$

$$G(\infty) = \infty \quad \text{if } c = 0$$

Also, we define $G(-d/c) = \infty$

We define

$T_b(z) = z + b$ translation

$J(z) = \frac{1}{z}$ inversion

and $M_a(z) = az$, respectively

We can write any LFT as

a composition these 3 maps:

In fact, we can write

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)} + \frac{a}{c}.$$

If $c=0$, then $\dfrac{az+b}{d} = \dfrac{a}{d}\left\{z+\dfrac{b}{a}\right\}$.

Every holomorphic function with non-vanishing derivative has the form $f(z) = (c+di)(x+yi)$

$$= (cx-dy) + i(cy+dx)$$

or in real variables

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Note angle between $\begin{pmatrix} c \\ -d \end{pmatrix}$ and

$\begin{pmatrix} d \\ c \end{pmatrix} = \frac{\pi}{2}$. Also, note that the vectors

$\begin{pmatrix} c \\ -d \end{pmatrix}$ and $\begin{pmatrix} d \\ c \end{pmatrix}$ have the same magnitude. Thus, if

$f'(z_0) \neq$, then the

linear descriptions is

a rotation and a dilation

Angles are preserved, and

all directions have same

magnitude.

The above composition

shows that any line or

Another property is

that if $G_1(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\dfrac{az+b}{cz+d}$

and $G_2(z) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $\dfrac{Az+B}{Cz+D}$

then $G_2 \circ G_1 =$

$$= \frac{\alpha z + \beta}{\gamma z + \delta}.$$

where $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

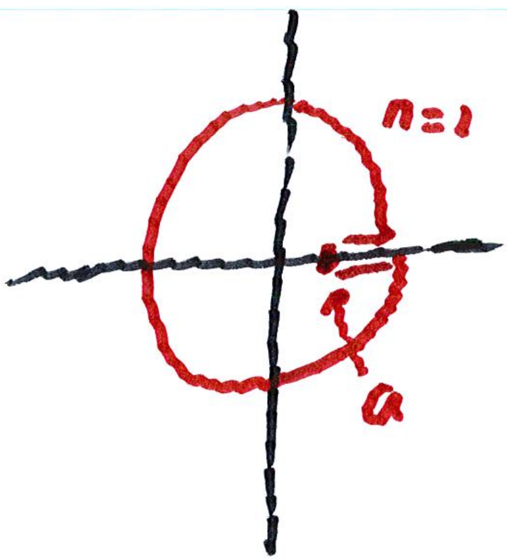
This is useful when we want to find the inverse LFT.

If $G_1(z) = \dfrac{az+b}{cz+d}$,

then $G_1^{-1} = \dfrac{dz-c}{-bz+a}$

One useful LFT sends the upper half plane to the disc.

Map the slit disk to the unit disk.



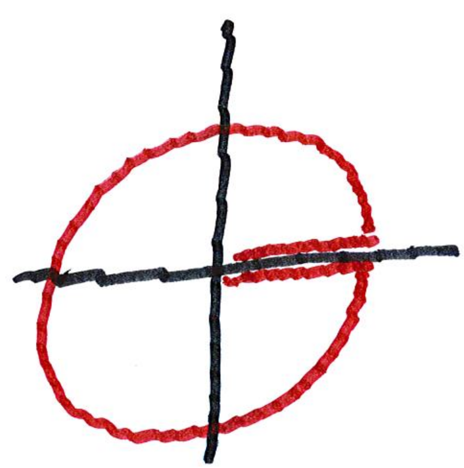

Now apply $\varphi_a$

$$= \frac{z-a}{1-\bar{a}z}$$

$$= \frac{z-a}{1-az}$$

Note the segment

$$[-1,1] \xrightarrow[\varphi_a]{} [-1,1]$$



Now apply $\sqrt{z}$ , $0 \le \theta < 2\pi$

$$\log z = \ln r + i\theta$$

so $\sqrt{z} = \exp\left(\frac{1}{2}\log z\right)$

$$= \exp\left\{\frac{1}{2}\ln r + \frac{i\theta}{2}\right\}$$

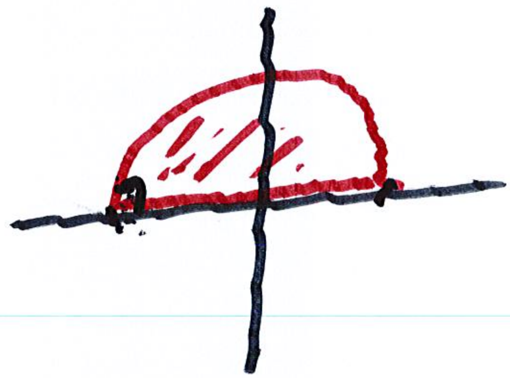
$$= r^{1/2} \cdot e^{\frac{i\theta}{2}}$$

$$= r^{1/2} \cdot e^{i\theta/2}$$

In our example $0 \le r \le 1$,

and slit $= 0$ or $\cancel{2\pi}$ $1$.

We get



We need an LFT with

$-1 \longrightarrow 0$

$1 \rightarrow \infty$

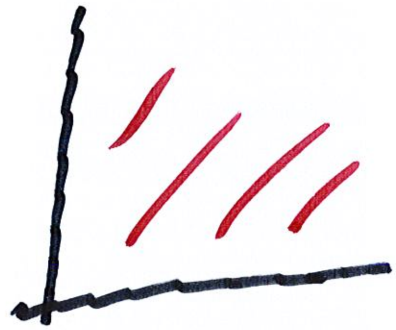
$0 \rightarrow 1$

$\dfrac{z+1}{z-1}$ But we have

$0 \rightarrow -1$. Mult. by $-1$.

$\therefore \; G(z) = \dfrac{z+1}{-z+1}$

We get



First Quadrant

Now apply $z^2$

Upper Half plane

Apply $G(z) = \dfrac{z-i}{z+i}$