Section 2.2

Holomorphic Functions

Def'n. Let z_0 be a point in an open set Ω. We say a function $f(z)$ in Ω is holomorphic at z_0 if

$$(1) \quad \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{exists.}$$

The key point is that h is a complex number that can approach 0 from <u>any</u> direction.

We call the above limit $f'(z_0)$ and say $f'(z_0)$ is the derivative of f at z_0

If $z = x + iy$, then we can write

$$f(z) = u(x,y) + iv(x,y).$$

If we also write $h = h_1 + ih_2$,

then the limit in (1) becomes

$$\lim_{h_1, h_2 \to 0} \frac{u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y)}{h}$$

Suppose that $h_2 = 0$ so that $h = h_1$

Then the above limit is:

$$\lim_{h \to 0} \frac{u(x+h, y) + iv(x+h, y) - u(x,y) - iv(x,y)}{h}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z_0)$$

Similarly, if $h_1 = 0$ so $h = ih_2$,

then the limit becomes

$$\lim_{h_2 \to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - u(x,y) - iv(x,y)}{ih_2}$$

$$= \frac{\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial y}}{i} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f'(z_0)$$

We conclude that

(2) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

and also that

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \text{and} \quad \frac{\partial v}{\partial y}$$

all exist at z_0. The equations in (2) are Cauchy-Riemann Eq'ns

The definition in (1) is equivalent to the statement that there is a complex number c so that

(3) $$f(z_0+h)-f(z_0)-ch = hE(h),$$

where $\lim_{h\to 0} E(h) = 0.$

Using (3), one can prove that

if f and g are holomorphic

in Ω, then

(i) $f+g$ is holorphic in Ω

and $(f+g)' = f' + g'$

(ii) fg is holorphic in Ω

and $(fg)' = f'g + fg'$

(iii) If $g(z_0) \neq 0$, then f/g is

holomorphic at z_0 and

$$\left(f/g\right)' = \frac{f'g - fg'}{g^2}.$$

Also, if $f : \Omega \to U$ and

$g : U \to \mathbb{C}$ are holorphic,

the Chain Rule holds:

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

for all $z \in \Omega$.

Note: Every polynomial $p(z)$

$$= \sum_{n=0}^{N} a_n z^n \text{ is holomorphic on } \mathbb{C}.$$

It is natural to go from polynomials to power series

We define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

When z is real, this coincides with the definition of e^x.

The complex series converges absolutely since $\left|\frac{z^n}{n!}\right| \leq \frac{|z|^n}{n!}$

and since $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$

converges.

Other examples are

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

and

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

These series are related by

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$+ i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

(by separating the terms

into even and odd powers of z)

$$e^{iz} = \cos z + i \sin z$$

Convergence of Power Series

Ex. Show $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

$$S_N(z) = 1 + z + z^2 + \cdots + z^N$$

$$(1-z)(1 + z + z^2 + \cdots + z^N)$$

$$= 1 - z^{N+1}$$

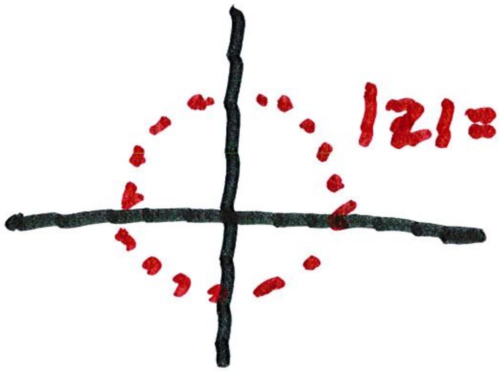
$$\therefore S_N(z) = \frac{1 - z^{N+1}}{1-z}$$

As $N \to \infty$ $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$,

provided $|z| < 1$.

Clearly $\{z^n\} \to \infty$ as

$n \to \infty$ if $|z| > 1$



$\therefore \sum_{n=0}^{\infty} z^n$ converges if $|z| < 1$.

Theorem. Given a series

$\sum_{n=0}^{\infty} a_n z^n$, there are three

3 alternatives

(i) $\sum_{n=0}^{\infty} a_n z^n$ converges

for all z, and absolutely

and uniformly on each

disk $D_R(0)$.

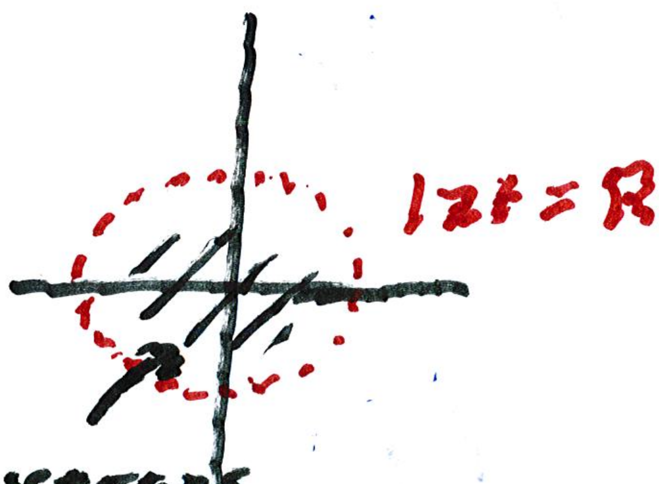
(ii) $\sum_{n=0}^{\infty} a_n z^n$ converges

only at 0

(iii) There is a positive

number R such that

$\sum_{n=0}^{\infty} a_n z^n$ converges uniformly and absolutely on each Disk $D_r(0)$ when $r < R$. Furthermore, the sequence $\{|a_n z^n|\}$ is unbounded if $|z| > R$.



Lemma.

Suppose that there is $w \neq 0$ such that

$\sum_{n=0}^{\infty} a_n w^n$ is convergent

Then $\{a_n w^n\}$ is bounded.

i.e., there is a $T > 0$ so that

$|a_n w^n| \leq T$, for all n.

Now suppose $r < |w|$.

Then, if $|z| \leq r$,

$$|a_n z^n| \le |a_n r^n|$$

$$\le |a_n w^n| \cdot \left(\frac{r}{|w|}\right)^n$$

$$\le T\rho^n \text{ for any } n.$$

where $\rho = \dfrac{r}{|w|} < 1$.

Since $\sum_{n=0}^{\infty} T\rho^n$ converges,

it follows that $\sum_{n=0}^{\infty} a_n z^n$

converges uniformly and absolutely on $D_r(0)$.

Now let

$$R = \sup\left\{ |w| ; \sum_{n=0}^{\infty} a_n w^n \text{ converges.} \right\}$$

If $\sum_{n=0}^{\infty} a_n w^n$ converges only when $w = 0$, this is Alternative (ii).

If $R = \infty$, then for any r with $0 < r < \infty$, there is a w with $r < |w|$

such that $\sum_{n=0}^{\infty} a_n w^n$ converges

Hence the lemma implies

$$\sum_{n=0}^{\infty} a_n z^n \text{ converges}$$

uniformly and absolutely

on D_r (Alternative (i))

If R is finite and positive,

then for any r with $0<r<R$

there is w with

$r < |w| < R$ such that

$$\sum_{n=0}^{\infty} a_n w^n \text{ converges.}$$

Again, the lemma implies

that Alternative (iii) holds.

Hadamard's Formula

Given a series as in the above theorem, the number R satisfies

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} .$$

If Alt. (i) holds, then for any $r < \infty$, the sequence $|a_n r^n|$ is bounded

Hence, for large n,

$$|a_n R^n| \le 1 .$$

$$\therefore |a_n|^{1/n} \cdot R \le 1 ,$$

or $|a_n|^{1/n} \le \frac{1}{R}$, large n

Letting R be arbitrarily large $\lim_{n \to \infty} |a_n|^{1/n} = 0 = \frac{1}{\infty}$

If Alt. (ii) holds,

$|a_n r^n|$ is unbounded

$\therefore$ For infinitely many n,

$|a_n r^n| \geq 1$

$\therefore$ $|a_n|^{1/n} \geq \frac{1}{r}$, inf. many n,

any $r > 0$

$\limsup |a_n|^{1/n} \geq \frac{1}{r}$

$\therefore \limsup\limits_{n\to\infty} = \infty = \frac{1}{0}$

Finally, suppose that

Alt. (iii) holds.

Choose R_1, R_2 so

$$R_1 < R < R_2$$

For large n,

$$|a_n| R_1^n \leq 1$$

$$\Rightarrow |a_n|^{1/n} \leq \frac{1}{R_1}$$

$$\therefore \limsup |a_n|^{1/n} \leq \frac{1}{R_1}$$

Also. $|a_n| R_2^n$ is unbounded.

for inf. many n.

$\therefore$

$\therefore \quad |a_n|^{1/n} R_2 \geq 1$

$\Rightarrow \quad |a_n|^{1/n} \geq \dfrac{1}{R_2}$ many n

$\Rightarrow \quad \limsup |a_n|^{1/n} \geq \dfrac{1}{R_2}$

$\therefore \quad \dfrac{1}{R_2} \leq \limsup\limits_{n \to \infty} \leq \dfrac{1}{R_1}$