Lesson 3

Thm 1. Given a series $\sum_{n=0}^{\infty} a_n z^n$,

we define R with $0 \le R \le \infty$

such that

$$R = \sup \left\{ |z| \text{ such that the sequence } n \to |a_n z^n| \text{ is unbounded} \right\} \quad (1)$$

There are 3 alternatives.

(i) If $R = 0$, then the sequence

only converges when $z = 0$.

(ii) If $R = \infty$, then the series converges absolutely on each disk $D_r(0)$ for every number $r > 0$.

(iii) If $0 < R < \infty$, then the series is unbounded for all z with $|z| > R$.

Moreover, for every r with $0 < r < R$, the series converges uniformly and absolutely on the disk $D_r(0)$.

Pf. Note that the series

$$\sum_{n=0}^{\infty} a_n z^n$$ converges when $z = 0$.

If (i) holds and $R = 0$, then the definition of R implies that the series that each sequence

$\{|a_n z^n|\}$ is unbounded

for every $z \neq 0$, which

implies that $\sum_{n=0}^{\infty} a_n z^n$

converges only when $z = 0$.

If (ii) holds and $R = \infty$,

let r be any finite positive

number. By the definition

of R, there must be a number z_1 with $|z_1| = r_1 > r$, such that $n \longrightarrow |a_n z_1^n| = |a_n| r_1^n$ is bounded. Hence there is $M > 0$ so that

$$|a_n| r_1^n \le M, \quad n = 0, 1, 2, \dots$$

Thus, if $|z| \le r$, then

$$|a_n z^n| \le |a_n r^n| = |a_n r_1^n| \left(\frac{r}{r_1}\right)^n$$

$$\le M \left(\frac{r}{r_1}\right)^n$$

Since the series $\sum_{n=0}^{\infty} M\left(\frac{r}{r_1}\right)^n$

converges $\left(r/r_1 < 1 \right)$,

we conclude that $\sum_{n=0}^{\infty} a_n z^n$

converges uniformly and

absolutely for all $z \in D_r(0)$.

In Case (iii), when

$0 < R < \infty$, it follows from (1)

that the sequence is unbounded

when $|z| > R$.

Now let r be any number with $0 < r < R$. There must be a numbe z_2 with $|z_2| = r_2 > r$ so that $\{|a_n| r_2^n\}$ is ~~un~~bounded. Here we get

$$|a_n| r_2^n \leq M, \quad n = 0, 1, 2, \ldots$$

It follows above that if $|z| \leq r$, then $|a_n z^n| \leq M \left(\frac{r}{r_2}\right)^n$.

Thus $\sum_{n=0}^{\infty} a_n z^n$ converges

absolutely and uniformly

on each $D_r(0)$ for all $r < R$.

In particular $\sum_{n=0}^{\infty} a_n z^n$ converges

on $D_R(0)$ and diverges

if $|z| > R$.

We now show that we can differentiate a power series.

Thm. Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Then

(i) The above series defines a holomorphic function $f(z)$ for all z in $D_R(0)$

(ii) The radius of convergence of ~~f(z)~~ the following series also equals R.

(iii) For all z in $D_R(0)$,

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Pf. Note that $\{a_n z^n\}$, $n=0,1,2,\ldots$

is unbounded if $|z| > R$.

Hence $\frac{n}{|z|} \{a_n z^n\}$ is

also unbounded if $|z| > R$.

If $|z| < R$, choose r so that $|z| < r < R$.

Then $|a_n r^n| \leq M$ for each n. Hence

$$|a_n z^n| \leq M \left(\frac{|z|}{r}\right)^n$$

Hence

$$n \to Mn \left(\frac{|z|}{r}\right)^n$$

also converges. This proves (ii)

Write $f(z) = S_N(z) + E_N(z)$,

where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \text{ and}$$

$$E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Suppose $z_0 \in D_R(0)$ and

choose r so $|z_0| < r < R$.

Let $g(z)$ (for $|z| < R$)

be the value of the

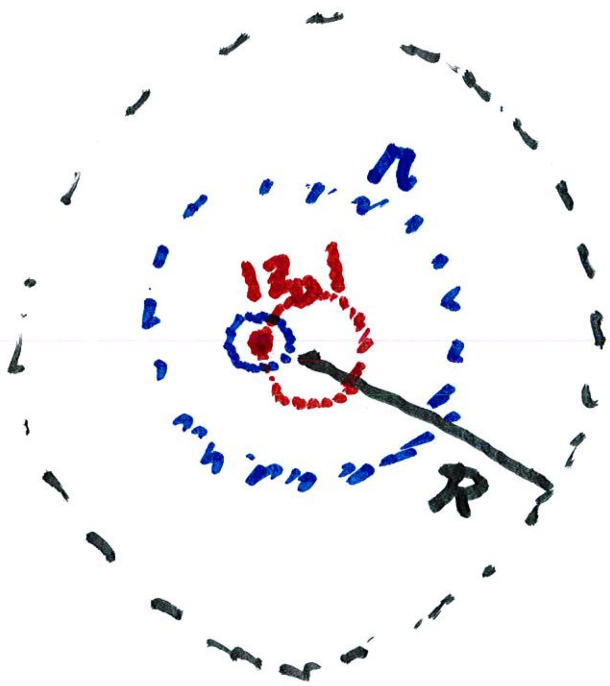
power series on the right

hand side of (iii).

We want to show that

f is differentiable in $D_R(0)$

and that $f'(z) = g(z)$.



We assume that h satisfies

$|z_0 + h| < r$.

We write the difference

quotient at z_0 so

$$\frac{f(z_0+h)-f(z_0)}{h}$$

$$= \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right)$$

$$+ \left(S_N'(z_0) - g(z_0) \right)$$

$$+ \left(\frac{E_N(z_0+h) - E_N(z_0)}{h} \right).$$

For term 3, we have the identity

$$(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$$

Hence,

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right|$$

where $a = z_0 + h$ and $b = z_0$

$$\leq \sum_{n=N+1}^{\infty} |a_n| \, n r^{n-1},$$

where we have used the fact that $|z_0| < R$ and $|z_0 + h| < R$

The series on the right is the tail of a convergent, since g converges absolutely on $|z| \le R$. Hence, we can choose N, sufficient large so that

$$\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| < \varepsilon.$$

For term 2, since $\lim_{N \to \infty} S_N'(z_0) = g(z_0)$

we can find N_2 so that

$N > N_2$ implies

$$|S_N'(z_0) - g'(z_0)| < \varepsilon .$$

Now fix $N > \mathrm{Max}\{N_1, N_2\}$.

Then choose δ suff. small

so if $0 < |h| < \delta$, then

$$\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right| < \varepsilon,$$

Since the error of a difference quotient converges to 0 as $h \to 0$.

Thus, if $0 < h < \delta$,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| < 3\varepsilon$$

This proves (i) and (iii).