An application of the differentiation theorem.

Recall we defined $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$

Theorem. For any z, w

$$e^{z+w} = e^z e^w.$$

Pf. First, we define

$$g(z) = e^z e^{-z}.$$

By the Chain Rule :

$g'(z) = e^z e^{-z} - e^z e^{-z} = 0$

$\therefore$ (by the HW problem)

$g(z) \equiv C$. Setting $z = 0$,

$g(0) = 1 \Rightarrow \therefore e^z e^{-z} = 1$

Now set (for fixed w)

$f(z) = e^{z+w} e^{-z} e^{-w}$.

By differentiating in z:

$$f'(z) = e^{z+w} e^{-z} e^{-w} - e^{z+w} e^{-z} e^{-w}$$

$= 0$ {using the Chain Rule}

so $f'(z) = 0$. $\Rightarrow f(z) \equiv C$.

Setting $z = 0$,

$e^{w} e^{-w} = 1$ (as above)

$\therefore\ e^{z+w} e^{-z} e^{-w} = 1$

Multiply by e^{w} and e^{z},

we get $e^{z+w} = e^{z} e^{w}$

Cauchy-Riemann Equations in Polar Coordinates.

Define $F(r, \theta) = f(r\cos\theta, r\sin\theta)$

Diff. wrt r. Then

$$F_r = \cos\theta\, f_x + \sin\theta\, f_y$$

By convention, we write

$F_r = f_r$. By the traditional

Cauchy-Riemann Eq'ns:

$f_y = i f_x$. Hence.

$$f_r = f_x \{\cos\theta + i \sin\theta\}$$

Now diff. wrt θ:

$$f_\theta = f_x \{-r\sin\theta\} + f_y \{r\cos\theta\}$$

The traditional Cauchy-Riemann

Eq'ns imply:

$$f_\theta = f_x \{-r\sin\theta + i r\cos\theta\}$$
$$= r i \{\cos\theta + i\sin\theta\} f_x.$$

Thus we have:

$$f_\theta = ri f_r .$$

Writing $f(r,\theta) = u(r,\theta) + i v(r,\theta)$

we get

$$u_\theta + i v_\theta = ri(u_r + i v_r)$$

$$\Rightarrow \quad u_\theta = -r v_r$$

and $\quad u_r = \frac{1}{r} v_\theta$

Ex. Show that

$\log z = \log r + i\theta$ is holomorphic

Note that $v_\theta \equiv 0$ and $v_r \equiv 0$.

and $u_r = \frac{1}{r}$ and $v_\theta = 1$

$\therefore \quad \frac{1}{r} = \frac{1}{r} \cdot 1 \quad \checkmark$

Integration along Curves

1. A __parametric curve__ is a continuous function from an interval $[a, b]$ to $\mathbb{C}$

2. The curve $z(t)$ is __smooth__ if $z'(t)$ is continuous and $z'(t) \neq 0$ for all $t \in [a, b]$

3. The curve γ is piecewise-smooth

if there are numbers

$a = a_0 < a_1, \dots, a_n = b$

where $z(t)$ is smooth on

each interval $[a_k, a_{k+1}]$.

4. Two parameterizations

$z : [a,b]$ and $\tilde{z} : [c,d] \to \mathbb{C}$

are <u>equivalent</u> if there is

a C^1 bijection $s \to t(s)$

such that $\tilde{z}(s) = z(t(s))$

5. We can define a curve γ^-

that reverses the orientation

by defining $z^-(t) = z(b+a-t)$

6. We say γ is <u>closed</u> if $z(a) = z(b)$.

7. We define the integral

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

where f is continuous along γ.

This definition is independent of the parameterization since

$$\int_a^b f(z(t))\, z'(t)\, dt = \int_c^d f(z(t(s)))\, z'(t(s))\, t'(s)\, ds$$

$$= \int_c^d f(\tilde{z}(s))\, \tilde{z}'(s)\, ds$$

8. The length of the smooth curve γ is

$$\text{length}(\gamma) = \int_a^b |z'(t)|\, dt$$

9. Two important properties are

(i) If γ^- is the reverse parameterization of γ,

$$\int_{\gamma^-} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

(ii)

$$\left|\int_{\gamma} f(z)\,dz\right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma)$$

10. Suppose f is a function on Ω. A <u>primitive</u> for f on Ω is a function F that is <u>holomorphic</u> on Ω and such that $F'(z) = f(z)$ on Ω.

11. If f is continuous and has a primitive F on Ω and if γ is a curve that

starts at w_1 and ends at w_2,

then $\displaystyle\int_\gamma f(z)\,dz = F(w_2) - F(w_1).$

Pf. $$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b F'(z(t))\,z'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left(F(z(t))\right)\cancel{z'(t)}\,dt$$

$$= F(z(b)) - F(z(a))$$

$$= F(w_2) - F(w_1)$$

If γ is only piecewise smooth,

then

$$\int_{\gamma} f(z)\,dz = \sum_{k=0}^{n-1} \int_{\gamma_k} f(z)\,dz$$

$$= \sum_{k=0}^{n-1} F(z(a_{k+1})) - F(z(a_k))$$

$$= F(z(a_n)) - F(z(a_0))$$

$$= F(w_2) - F(w_1)$$

Corollary: If γ is a closed curve and f is continuous and has a primitive in Ω, then

$$\int_\gamma f(z)\,dz = 0.$$

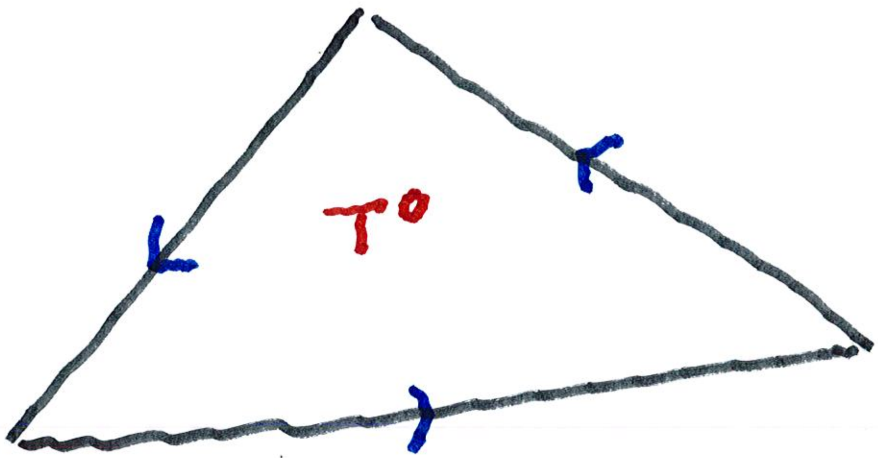
Chapter 2. Cauchy's Theorem

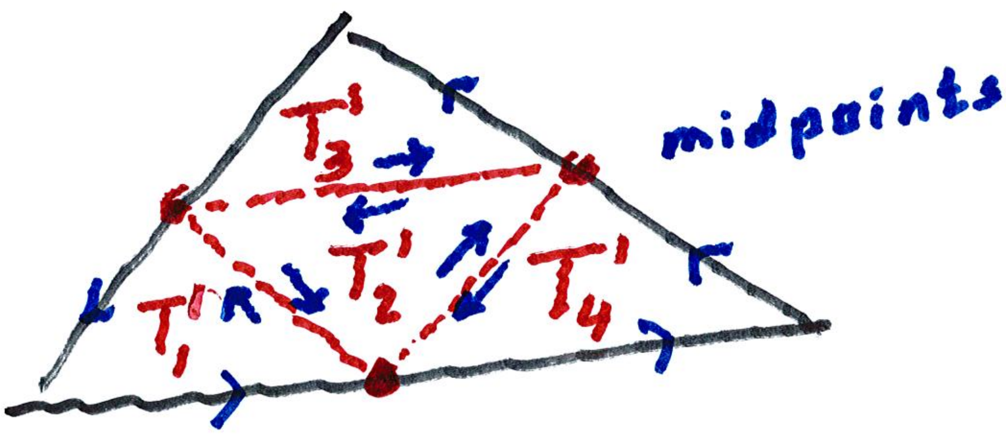
Special Case:

Goursat's Theorem. If Ω is an open set in $\mathbb{C}$ and if $T \subset \Omega$ is a triangle whose interior is also in Ω, then if f is holomorphic in Ω,

$$\int_T f(z)\,dz = 0$$

Pf.





All triangles have the same (counter-clockwise) orientation

Note that

$$\int_{T^0} f(z)\,dz = \int_{T_1^1} f(z)\,dz + \int_{T_2^1} f(z)\,dz$$

$$+ \int_{T_3^1} f(z)\,dz + \int_{T_4^1} f(z)\,dz$$

Choose $j = 1, 2, 3,$ or 4 so

$\left| \int_{T_j^1} f(z)\,dz \right|$ is <u>largest</u>

among $j = 1, 2, 3,$ or 4

Choose $j \in \{1, \dots, 4\}$ so

$$\left| \int_{T^0} f(z)\,dz \right| \leq 4 \left| \int_{T_j^1} f(z)\,dz \right|$$

Now relabel T_j^1 as T^1

Note that if d^1 and p^1 are the diameter and perimeter, then

$$d^1 = \frac{1}{2} d^0$$

$$p^1 = \frac{1}{2} p^0 .$$