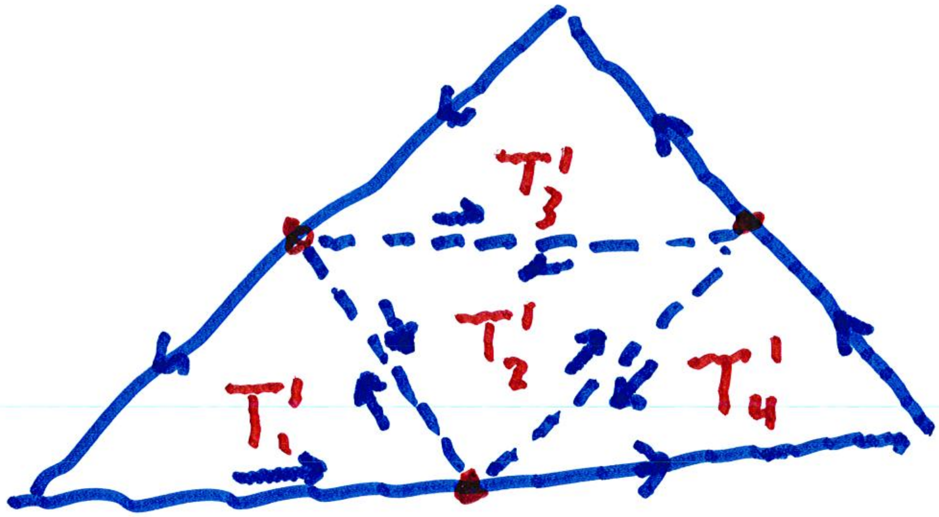
Recall f is holomorphic in a neighbord Ω of a triangle T_0 (including the interior)

We define triangles

T_1^1, T_2^1, T_3^1, and T_4^1

by



We assume all triangles have the (counter-clockwise) same orientation. Then

$$\int_{T^0} f(z)\,dz = \int_{T_1'} f(z)\,dz + \int_{T_2'} f(z)\,dz + \int_{T_3'} f(z)\,dz + \int_{T_4'} f(z)\,dz$$

Choose $j = j_1$ so that

$\left| \int_{j_1} f(z)\,dz \right|$ has the largest

value (among all j, $1 \le j \le 4$).

Then $$\left| \int_{T^0} f(z)\,dz \right| \le 4 \left| \int_{T^{j_1}} f(z)\,dz \right|$$

Now set $T^1 = T^1_{j_1}$

Note that if d_k and p_k

= diameter and perimeter,

resp. of T^k, $k=0,1$,

then $d_1 = \frac{1}{2} d_0$ and $p_1 = \frac{1}{2} p_0$.

Continuing this process

we obtain a sequence of

triangles $T^0, T^1, \dots T^n, \dots$

so that

(1) $T^0 \supset T^1 \supset T^2 \supset \dots \supset T^n \supset \dots$

and $\left| \int_{T^0} f(z)\,dz \right| \le 4^n \left| \int_{T^n} f(z)\,dz \right|$

and $d^n = 2^{-n} d^0, \quad p^n = 2^{-n} p^0.$

Assuming T^n = solid triangle

so that (1) still holds,

we see that there is a point z_0 so that $z_0 \in T^n$ for every n.

Since f is holomorphic near z_0, there is a function $\Psi(z)$ with

$$\lim_{z \to z_0} \Psi(z) = 0$$

and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \Psi(z)(z - z_0)$$

Note that

$$F(z) := f(0)z + f'(z_0)\frac{(z-z_0)^2}{2}$$

is a primitive of $f(z_0) + f'(z_0)(z-z_0)$

Hence,

$$\int_{T^n} f(z)\,dz = \int_{T^n} \psi(z)(z-z_0)\,dz. \qquad (2)$$

Now let $\varepsilon_n = \sup\limits_{z \in T_n} |\psi(z)|$

Since $|z - z_0| \le d_n$, we obtain that

$$\left| \int_{T^n} f(z)\,dz \right| \le \varepsilon_n \, d_n \, p_n$$

$$\le \varepsilon_n \, 4^{-n} \, d_0 \, p_0 .$$

Hence

$$\left| \int_{T^0} f(z)\,dz \right| \le 4^n \left| \int_{T^n} f(z)\,dz \right| \le \varepsilon_n \, d_0 \, p_0$$

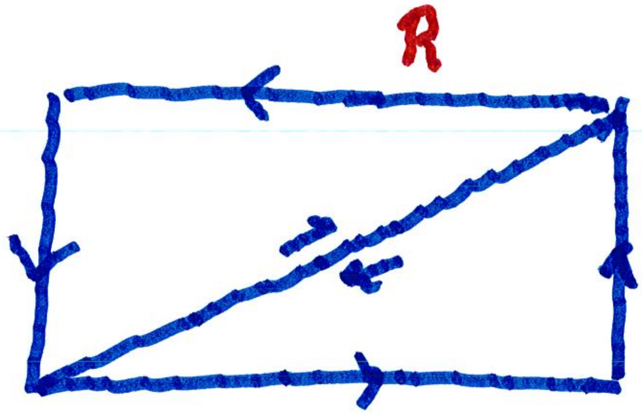
$\to 0$ as $n \to \infty$

Corollary of Goursat's Thm.

Suppose f is holomorphic in an open set Ω and if a rectangle R and its interior are contained in Ω, then

$$\int_R f(z)\,dz = 0.$$

Pf. This follows from



Local Existence of

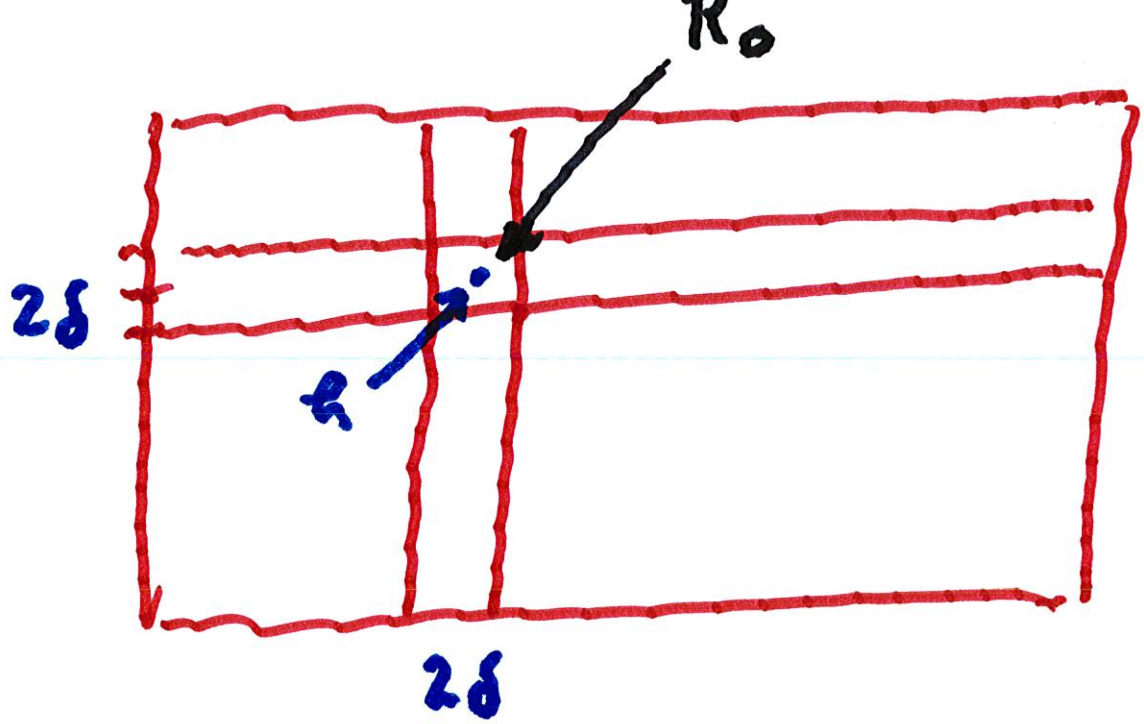
Primitives

Thm. A holomorphic function on an open disk has a primitive on that disk.

Theorem. Let $\overset{f}{R}$ be holomorphic in a neighborhood of a rectangle except at a finite number of points z_j, $j=1,\dots,N$. Then $\int_{\partial R} f(z)\,dz = 0$ where ∂R is the boundary of R. Also assume $|f(z)|$ is bounded near z_j

We first assume that there is only 1 point t_1.

Then we subdivide R into 9 rectangles, such that square contains t_1 at its center and has side length $= 2\delta$.



Then $\displaystyle\int_{\partial R} f(z)\,dz = \int_{\partial R_0} f(z)\,dz$.

We have

$$\left| \int_{\partial R_0} f(z)\,dz \right| \leq 8M\delta .$$

As $\delta \to 0$, we obtain that

$$\int_{\partial R} f(z)\,dz = 0.$$

In general, we can construct

N rectangles R_j, so ζ_j is

in the interior of R_j.

Construct the suitable

subdivision of R.

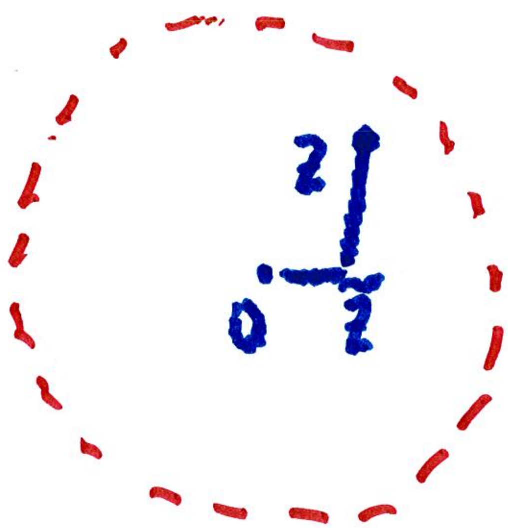
Pf. We can assume the disk

D is centered at the origin.

Let γ_z be the path that

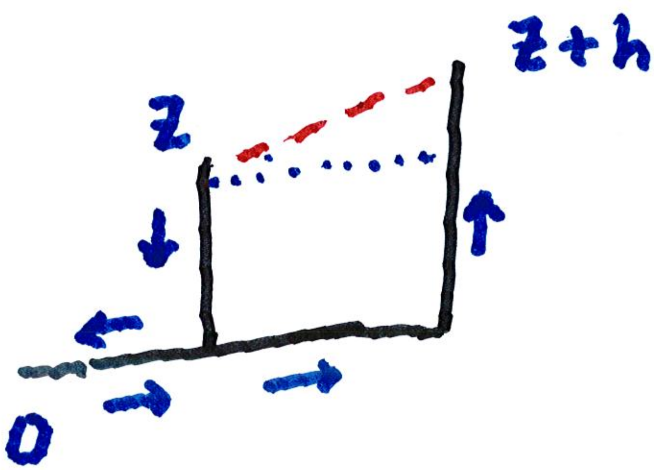
goes from 0 to $\tilde{z}$ and then

from $\tilde{z}$ to z, where $\operatorname{Re}\tilde{z} = \operatorname{Re} z$.



Similarly, γ_{z+h} is the analogous path from 0 to $z+h$.

This means $\gamma_{z+h} - \gamma_z$ is

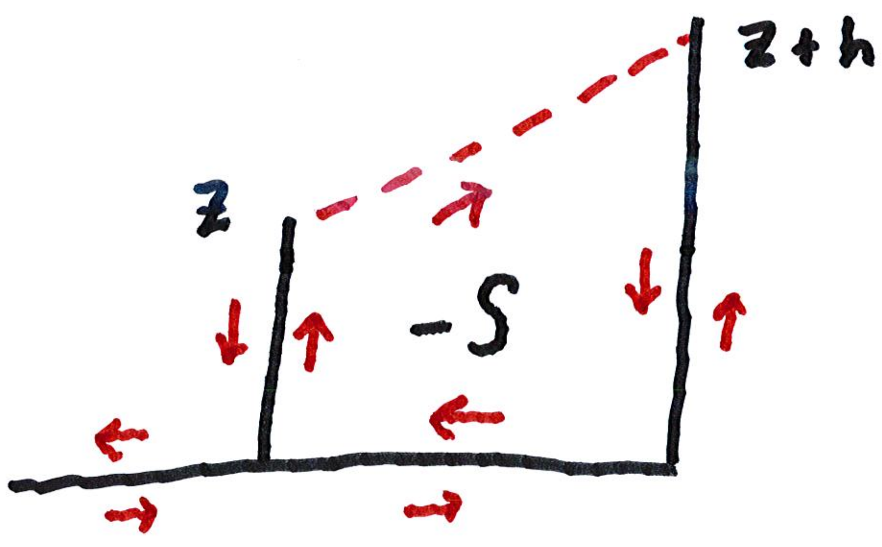


Let S be the counter-clockwise indicated trapezoid.

Then

$$\gamma_{z+h} - \gamma_z - S = \eta,$$

where η is the straight path from z to $z+h$.



Observe $\int_S f(z)\,dz = 0.$

We define $F(z) = \int_{\gamma_z} f(z)\,dz$

We conclude that

$$F(z+h) - F(z) = \int_{\eta} f(w)\,dw$$

where η is the straight path from z to $z+h$.

Recall $f(w) = f(z) + \varphi(w)$

where $\varphi(w) \to 0$ as $w \to z$.

(because f is continuous)

Hence, $F(z+h) - F(z)$

$$= \int_{\eta} f(z)\, dw + \int_{\eta} \varphi(w)\, dw$$

$$= f(z) \int_{\eta} dw + \int_{\eta} \varphi(w)\, dw$$

The first integral

is $f(z)(z+h-z) = f(z)h$.

The second satisfies

$$\left| \int_{\eta} \varphi(w)\, dw \right| \leq \sup_{w \in \eta} |\varphi(w)|\, |h|$$

Since $\lim |\varphi(w)| = 0$,

after dividing by h, we obtain

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$

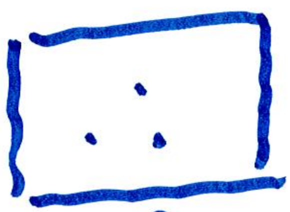
Hence, F is a primitive for f on the disk.

Thm. Suppose f is holomorphic on an open disk. Then

$$\int_\gamma f(z)\,dz = 0 \tag{3}$$

for any closed curve γ in that disk. In particular if γ is a circle so that C and its interior are in the open set, then $\int_C f(z)\,dz = 0$.

Pf. This follows since f has a primitive F in a neighborhood of γ and C.



R

$$\int_{\partial R} f(z)\,dz = 0$$

$$\int_{\gamma} f(z)\,dz = 0$$

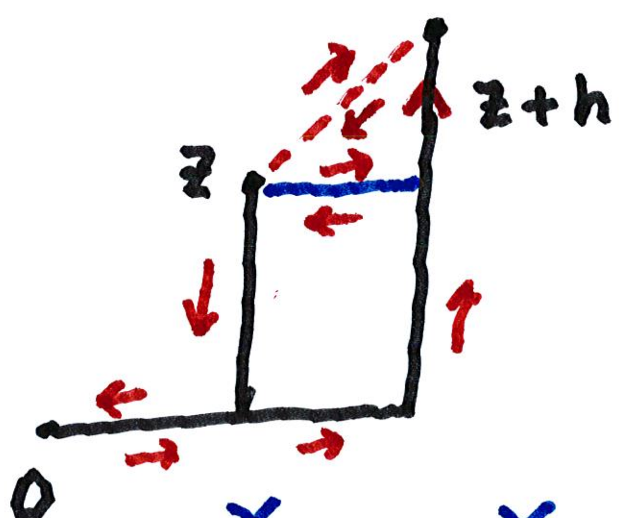
$$\gamma \subset \square \subset \Omega$$

Similarly γ_{z+h} is the

analogous path from 0 to $z+h$.

Let S be the trapezoid

as indicated. Then we have



$$\gamma_{z+h} - \gamma_z - S = \eta$$