# Lesson 6 (cleaned up)

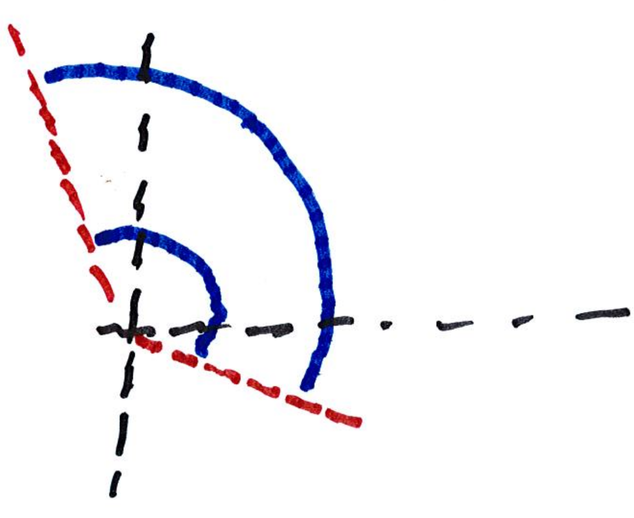
We say a domain $\Omega$ is star-shaped

if whenever $z \in \Omega$, then

$rz \in \Omega$, for $0 \leq r \leq 1$.

Ex. Let $\Omega = \left\{ z;\ R_1 < |z| < R_2 \right.$

$\left. \text{and } \frac{-\pi}{8} < \operatorname{Arg} z < \frac{5\pi}{8} \right\}$



$\Omega$ is star-shapet

if $R_2/R_1$ is large.

We also say $\Omega$ is star-shaped with respect to $z_0 \in \Omega$ if whenever $z \in \Omega$, then

$$r(z-z_0) + z_0 \in \Omega.$$

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Suppose that $f$ is holomorphic in an neighborhood $\Omega$ containing a triangle $\overline{T}$, except perhaps at one point $\zeta$.

If $\zeta$ does not lie in $\overline{T}$,

then Goursat's Theorem

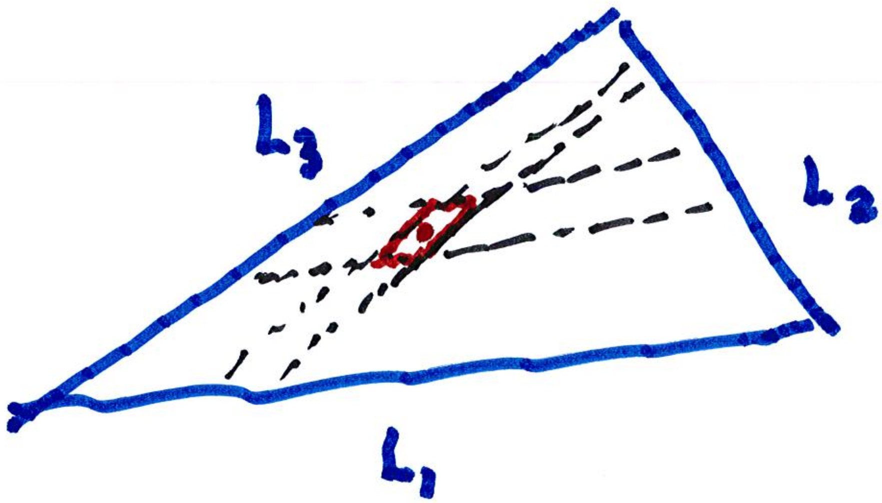
for triangles states:

$$\int_T f(z)\,dz = 0.$$

Now suppose that $\zeta$ lies in

the interior of $T$.

We decompose $\overline{\overline{T}}$ as follows:




We choose the pairs of lines to be parallel to $L_1$ and $L_3$ respectively.

We also need each pair to

be very close. We assume that each quadrilateral is decomposed into a pair of triangles, and that each triangle is given the positive rotation. Assuming WLOG that $\xi$ lies on only one of the triangles, say $T_N$.

Adding up all of the line

integrals (including $T^*$)

If we let $\tilde{T}$ be the original

triangle $L_1 + L_2 + L_3$, we get

$$\tilde{T} = T_1 + \cdots + T_{N-1} + T_N.$$

Goursat's Theorem implies that

$$\tilde{T} = 0 + \cdots + 0 + T_N.$$

If we also assume that

the perimeter of $T_N$ is bounded by an arbitrarily small number $p$, then $|T_N| \le pM < \varepsilon$.
which proves Goursat's Thm in this case.

Another case is when $z_0$ lies on one of the segments of $\tilde{T}$. We can modify the case when $z_0$ lies in int. of $T$.

Assuming this to be done, we conclude that $\int_{\tilde{T}} f(z)\,dz = 0$.

Now suppose that $\Omega$ is a star-shaped domain with (WLOG) respect to $0 \in \mathbb{C}$. We want to show that any holomorphic function $f$ on $\Omega$ has a primitive $F$

Let $z$ be a point in $\Omega$

For any straight line path

$P_z$ from $0$ to $z$, we define

$$F(z) = \int_{P_z} f(z)\,dz.$$

Similarly, for any small $h \in \Omega$,

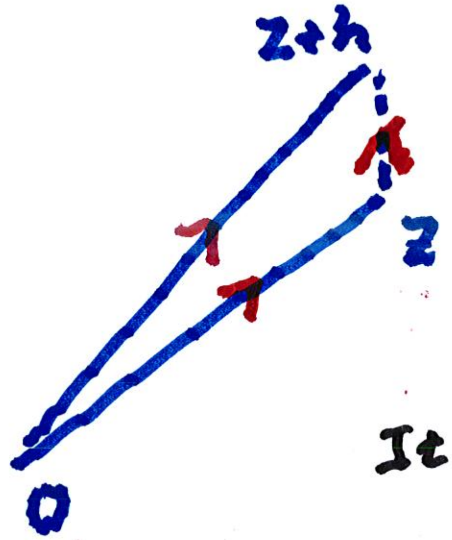
we define

$$F(z+h) = \int_{P_{z+h}} f(z)\,dz,$$

where $P_{z+h}$ is the straight path

from 0 to $z+h$. Letting $\eta$ be the straight from $z$ to $z+h$.

The orientations are given by the following diagram:




It follows that

$$\int_{P_{z+h}} f(z)\,dz = \int_{P_z} f(z)\,dz + \int_{\eta} f(w)\,dw$$

or

$$F(z+h) - F(z) = \int_{\eta} f(w)\,d$$

Since f is continuous at z,

we have $f(w) = f(z) + \psi(w)$,

where $\lim_{w \to z} \psi(w) = 0$.

Integrating along $\eta$, we get

$$\int_{\eta} f(w)\,dw = f(z)h + \int_{\eta} \psi(w)\,dw$$

The final integral is

$$\left| \int_\eta \psi(w)\,dw \right| \le \varepsilon|h|, \qquad (1)$$

since $\psi(w) \to 0$ as $w \to z$.

Dividing by $h$, we obtain

$$\frac{F(z+h) - F(z)}{h} = f(z) + \frac{1}{h}\int_\eta \psi(w)\,dw$$

It follows from (1) that

$$F'(z) = f(z).$$

Thus, we have proved:

Thm If $f$ is holomorphic on a star-shaped domain $\Omega$, then there is a holomorphic function $F$ on $\Omega$ so that $F'(z) = f(z)$ for all $z$ in $\Omega$.

Corollary: If $\gamma$ is any <u>closed</u> piecewise differentiable curve on a star-shaped domain, then $\int_\gamma f(z)\,dz = 0.$

We now derive the Cauchy Integral Formula:

Suppose that $f$ is holomorphic in an open set containing $\bar{D}_R$, the closed disk of radius $R$.

Then if $z$ lies in the interior of $\bar{D}_R$, then

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Pf. Set $F(\zeta) = \dfrac{f(\zeta) - f(z)}{\zeta - z}$

Note that $F(\zeta)$ is holomorphic

near $\bar{D}_R$ except at $z$.

We let $A_1$, $A_2$, $A_3$ and $A_4$

be the 4 paths depicted as

follows:

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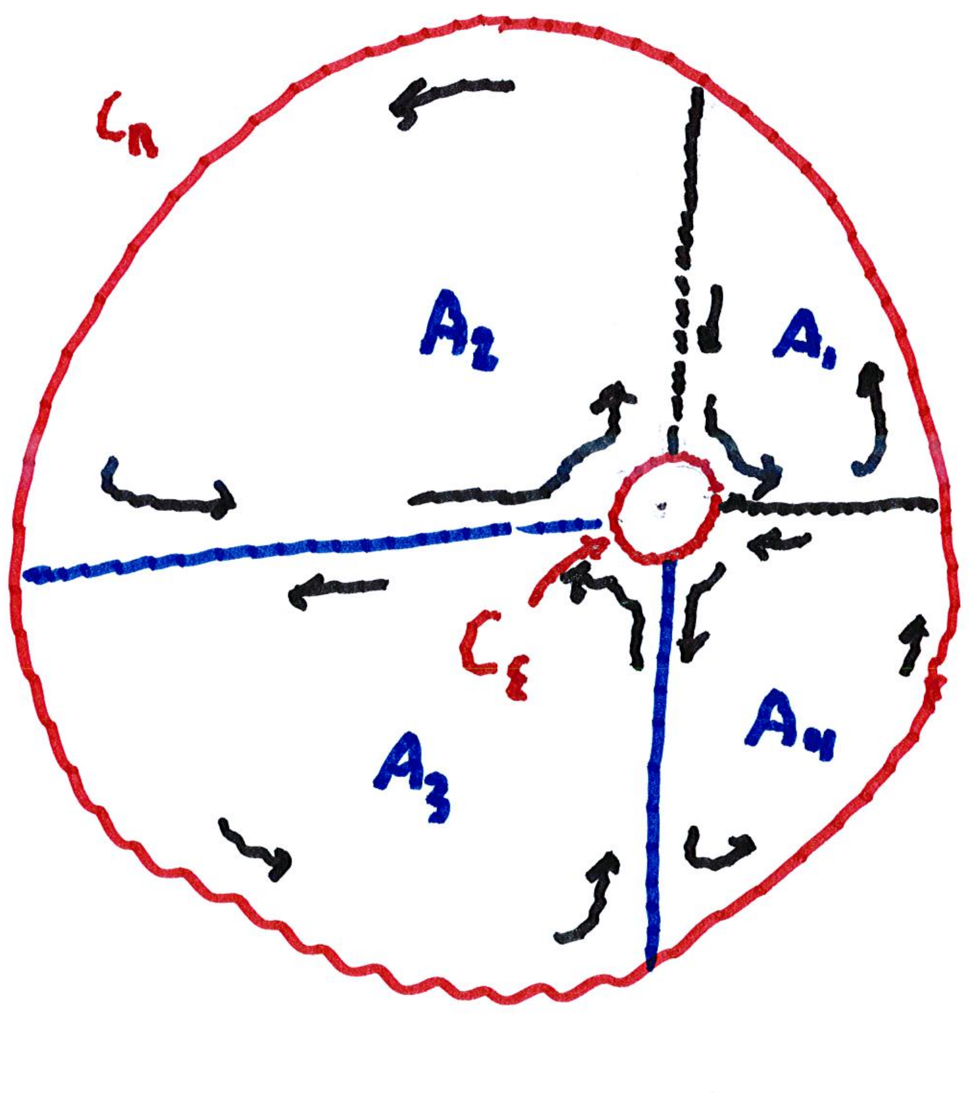


Since the integrals cancel on the line segments, we obtain

$$\int_{C_R} \frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta - \int_{C_\varepsilon} \frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta = 0.$$

Note that $\left|\frac{f(\zeta)-f(z)}{\zeta - z}\right| \leq M$

for $M$ independant of $\varepsilon$.

Taking the limit as ~~ne~~, $\varepsilon \to 0$

we obtain

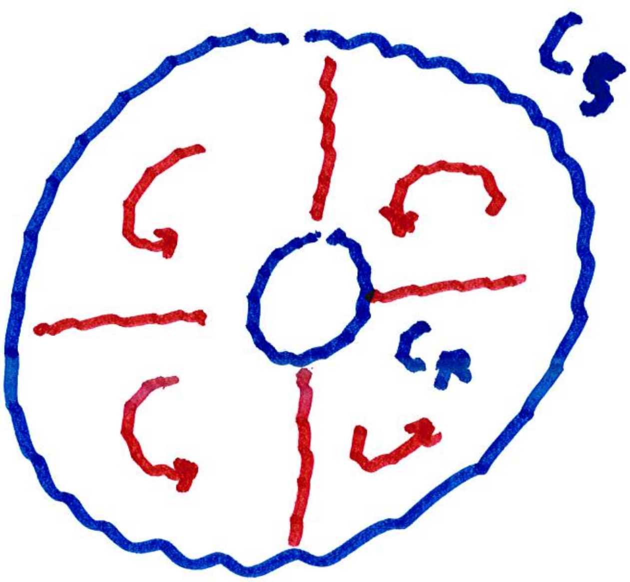
$$\int_{C_R} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0.$$

or

$$\int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{C_R} \frac{f(z)\, d\zeta}{\zeta - z}$$

We must compute $\displaystyle\int_{C_R} \frac{dx}{\zeta - z}$

Let $S > R$. If we decompose the area between $C_R$ and $C_S$, we obtain




$$\int_{C_S} \frac{d\zeta}{\zeta - z} = \int_{C_R} \frac{d\zeta}{\zeta - z}$$

If for the integral on $C_\delta$,

it can be parameterized as

follows: $z(t) = \delta e^{it}$.

$$\therefore \int_{C_\delta} \frac{d\zeta}{\zeta - z} = \int_0^{2\pi} \frac{i\delta e^{it}\, dt}{\delta e^{it} - z}$$

$$= \int_0^{2\pi} \frac{i\, dt}{e^{it} - \frac{z}{\delta}} \to \int_0^{2\pi} i\, dt = 2\pi i$$

Dividing by $2\pi i$, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$