Riemann Mapping (cont.)

We defined

$$\mathcal{F} = \{ f : \Omega_1 \to \mathbb{D} \text{, that is holomorphic, injective, and } f(0) = 0 \}$$

We also showed that we can assume that ~~we can assume that~~ Ω_1 is an open subset of $\mathbb{D}$

that contains 0.

we then showed that

$|f'(0)| \leq n$, for some $n > 1$

and that if we define

$$s = \sup_{f \in \mathcal{F}} |f'(0)|,$$

then $1 \leq s \leq n$. Using Montel's

Thm, we showed there is

a function $f \in \mathcal{F}$ so that

$|f'(0)| = s$.

It still remains to show that f is surjective. If not, there is $\alpha \in D$ such that

$\alpha \neq f(z)$ for all $z \in \Omega_1$.

We defined $\psi_\alpha = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$.

One easily shows that

$$\Psi_\alpha \circ \Psi_\alpha = \mathrm{Id} = z.$$

and that $\Psi_\alpha(0) = \alpha$, $\Psi_\alpha(\alpha) = 0$.

Since Ω is simply-connected, the open set $U = (\Psi_\alpha \circ f)(\Omega)$ is also simp.-connected. Also U does not contain 0.

Hence we can define a

square root by $g(w) = e^{\frac{1}{2}\log w}$.

on U.

Next consider the fcn.

$$F = \Psi_{g(\alpha)} \circ g \circ \Psi_\alpha \circ f$$

All components are 1-to-1

(If $g(z_1) = g(z_2) \to z_1 = z_2$ by squaring)

Also, $F : \Omega_1 \to \mathbb{D}$

since $\psi_\beta : \mathbb{D} \to \mathbb{D}$ for all β.

Also $g : \mathbb{D} \to \mathbb{D}$.

If $h(w) = w^2$, then

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$$

But Φ maps $\mathbb{D} \to \mathbb{D}$ with

$\Phi(0) = 0$, and is not injective

because F is and h is not.

By the ~~large~~ last part of

the Schwarz lemma, $|\Phi'(0)| < 1$.

By the Chain Rule

$$f'(0) = \Phi'(0)\, F'(0),$$

and so $|f'(0)| < |F'(0)|$

which contradicts maximality

of $|f'(0)|$ in $\mathcal{F}$.

We conclude that f maps Ω_1 onto D, which completes the proof.

Chapter 6. The gamma function

When $s>0$, we define

$$\Gamma(s)=\int_0^\infty e^{-t}\,t^{s-1}\,dt.$$

The function $e^{-t}t^{s-1}$ is

integrable at ∞, because

$e^{-t}t^p$ converges rapidly

at ∞. Also t^p is integrable

near 0 if $p > -1$.

Proposition. The gamma fcn

extends to an analytic function

$\mathrm{Re}\, s > 0$ (we use $s = \sigma + it$)

It is still given by the

integral

Pf. We show that the

integral defines a holomorphic

function in every strip

$$S_{\delta,M} = \{ \delta < \mathrm{Re}(s) < M \}$$

If σ = real part of s,

where $0 < \delta < M < \infty$

then $\left| e^{-t} t^{s-1} \right| = \left| e^{-t} t^{\sigma-1} \right|$

$$\left\{ \left| t^{(\sigma-1+i\tau} \right| = \left| t^{\sigma-1} \right| \right\}$$

so that the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$$

which is defined by the limit

$$\lim_{\varepsilon \to 0} \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1}\, dt$$

converges for each $s \in S_{\delta, M}$

For $\varepsilon > 0$, let

$$F_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1} \, dt .$$

The fcn. F_ε is holomorphic *

in the strip $S_{\delta, M}$. By 5.2

it suffices to show that

* Look at Thm. 5.4 in Chapter 2.

it suffices to show that

F_ε converges uniformly to

Γ on the strip $S_{\delta, M}$. To see this,

we first observe that

$$|\Gamma(s) - F_\varepsilon(s)| \leq \int_0^\varepsilon e^{-t} t^{s-1}\,dt + \int_{1/\varepsilon}^\infty e^{-t} t^{s-1}\,dt$$

One can easily show that both integrals converge uniformly to 0.

Lemma. If $\operatorname{Re} s > 0$, then

$$\Gamma(s+1) = s\Gamma(s).$$

This implies that

$$\Gamma(n+1) = n\Gamma(n), \text{ i.e.}$$

$$\Gamma(n+1) = n! \text{ for } n = 0, 1, 2, \ldots$$

Proof:

$$\int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{d}{dt}\left(e^{-t} t^{s}\right) dt = -\int_{\varepsilon}^{1/\varepsilon} e^{-t} t^{s}\, dt$$

$$+ s \int_{\varepsilon}^{1/\varepsilon} e^{-t} t^{s-1}$$

$$\shortparallel$$

$$e^{-\frac{1}{\varepsilon}} \frac{1}{\varepsilon^{s}}$$

$$- e^{-\varepsilon} \varepsilon^{s} \qquad \text{If } \varepsilon \to 0$$

$$s\,\Gamma(s) = \Gamma(s+1)$$

$$0 = s\,\Gamma(s) - \Gamma(s+1)$$

Thm. The fcn $\Gamma(s)$ initially defined for $\mathrm{Re}(s) > 0$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at the negative integers $s = 0, -1, -2, \ldots$

The residue of Γ at $s = -n$ is $\dfrac{(-1)^n}{n!}$

For $\mathrm{Re}(s) > 0$:

$$\Gamma(s) = \int_1^\infty e^{-t} t^{s-1} dt + \int_0^1 e^{-t} t^{s-1}$$

$$= \int_1^\infty e^{-t} t^{s-1} dt + \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+s)}$$

The sum defines a meromorphic fcn with poles at $s = -n$

For fixed $R > 0$, we may split the sum as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} = \sum_{n=0}^{N} \frac{(-1)^n}{n!(n+s)}$$

$$+ \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!(n+s)}$$

so that $N > 2R$. The first finite sum defines a meromorphic fcn in the disk $|z| < R$ with poles at the desired points. The second sum converges uniformly in in that disk. Hence, it defines a holomorphic fcn. there.

since $\left| \frac{(-1)^n}{n!(n+s)} \right| \le \frac{1}{n! R}$.

Since R is arbitrary, we conclude

that the series has the desired

properties: