

Math 530
Homework 2

1. Suppose that $\varphi(z)$ is a continuous function on the trace of a path γ . Prove that the function

$$f(z) = \int_{\gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta$$

is analytic on $\mathbb{C} - \text{tr } \gamma$.

2. If $U = \sum_{n=0}^{\infty} u_n$ and $V = \sum_{n=0}^{\infty} v_n$ are given by the sum of absolutely convergent series, show that $UV = \sum_{n=0}^{\infty} p_n$ where $p_n = \sum_{k=0}^n u_k v_{n-k}$ and that this sum converges absolutely.
3. Use the result of the previous problem to give a proof of the formula $\exp(z + w) = \exp(z) \cdot \exp(w)$. (You will need to use the binomial theorem.)
4. Suppose that an analytic function is written in polar form

$$f(re^{i\theta}) = U(r, \theta) + iV(r, \theta).$$

Derive the polar form of the Cauchy-Riemann equations,

$$rU_r = V_{\theta} \quad \text{and} \quad U_{\theta} = -rV_r.$$

Prove that if U and V are continuously differentiable and satisfy the polar Cauchy-Riemann equations on some polar rectangle, then $f(re^{i\theta}) = U(r, \theta) + iV(r, \theta)$ defines an analytic function there. Use your result to verify that the function $\text{Log}(re^{i\theta}) = \text{Ln } r + i\theta$ is analytic on $\{re^{i\theta} : r > 0, -\pi < \theta < \pi\}$.

5. Suppose that γ is a curve in the complex plane parameterized by a function $z(t)$ which is a continuously differentiable function from the interval $[a, b]$ into \mathbb{C} . Show that if f is analytic on an open set containing γ , then $w(t) = f(z(t))$ is differentiable on $[a, b]$ and $w'(t) = f'(z(t))z'(t)$. (This is another important Chain Rule.)
6. Use the result of the previous problem to prove that $\int_{\gamma} z^n dz = 0$ for any *closed* path γ and integer $n \neq -1$, assuming that $\text{tr } \gamma$ does not contain the origin if $n < 0$.