Math 530

Homework 3

1. Suppose that a_n is a sequence of non-zero complex numbers. Show that if

$$L = \lim_{n\to\infty} |a_n|/|a_{n+1}|$$

exists, then L is equal to the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Find an example of a sequence of non-zero terms a_n such that this limit fails to exist, and yet $\limsup_{n\to\infty} |a_n|^{1/n}$ is equal to one, and hence the associated power series has radius of convergence equal to one.

2. Suppose that $f(z)$ and $g(z)$ are given by convergent power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, respectively, where $R_f > 0$ and $R_g > 0$. Prove that, if $g(0) \neq 0$, then f/g is analytic in a neighborhood of the origin and the power series for $f(z)/g(z)$ is $\sum_{n=0}^{\infty} c_n z^n$ where the c_n's can be determined recursively via the formula,

$$a_n = \sum_{k=0}^{n} b_k c_{n-k}. \qquad (*)$$

Is the radius of convergence of this series at least as big as the minimum of the radii of convergence for the series for f and g? Can the radius be larger than this?

3. Use formula $(*)$ of problem 2 and the complex power series for sine and cosine to find the first few terms in the power series expansion for $\tan z = \sin z/\cos z$ about $z = 0$. What is the radius of convergence of the Taylor series for $\tan z$ about $z = 0$?

4. (Recall that a *domain* is an open connected set). It can be shown that an analytic function $f(z)$ on a domain Ω must be constant if **A)** f is *real valued* on Ω, or **B)** $|f|$ is constant on Ω, or **C)** $\arg f$ is constant on Ω. Instead of giving separate proofs for A-C, do the follwing single problem that implies them all. Suppose a curve Γ in the complex plane is described as the level set of a function ρ:

$$\Gamma = \{z = x + iy \in \mathbb{C} \,:\, \rho(x, y) = 0\},$$

where ρ is a real valued twice continuously differentiable function on $\mathbb{R}^2$ and $\nabla\rho$ is non-vanishing on Γ. Prove that if $f(z)$ is an analytic function on a domain Ω such that $f(\Omega) \subset \Gamma$, then f must be constant on Ω.

5. We know that $\exp(x + iy) = e^x(\cos y + i \sin y)$. Find a similar formula for $\sin z = z - z^3/3! + z^5/5! + \cdots$

6. Show that $\int_{-\infty}^{\infty} e^{-t^2} \cos 2bt \, dt = \sqrt{\pi} e^{-b^2}$ by integrating e^{-z^2} around the rectangle with corners at $\pm a$ and $\pm a + ib$. Let $a \to \infty$.