Math 530

Homework 4

1. For what values of z is the series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$ convergent? Same question for $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{1+z^{2n}}$.

2. If f is analytic on the unit disc and $|f(z)| \le 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that the Cauchy Estimates will yield.

3. Show that the successive derivatives of an analytic function at a point a can never satisfy $|f^{(n)}(a)| > n!n^n$. Formulate a sharper theorem of the same kind.

4. Prove that

$$\cos(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$$

without mentioning trigonometry or angles.

5. Suppose that f is analytic on a disk $D_\epsilon(0)$ and satisfies the differential equation $f'' = f$. Prove that f is given by $A\cosh z + B\sinh z$, where A and B are constants.

6. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

7. Prove that an entire function f such that $\operatorname{Re} f(z) > 0$ for all z must be constant.

8. Prove that there is no analytic function f on the unit disk such that $f(1/n) = 2^{-n}$ for $n = 2, 3, 4, \ldots$

9. Give a proof of the Fundamental Theorem of Algebra based on the Maximum Principle.

10. Show that $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$ by integrating e^{-z^2} around the counterclockwise boundary of $\{z = re^{i\theta} : 0 < r < R, 0 < \theta < \pi/4\}$ and letting $R \to \infty$.