Math 530

Homework 6

1. Use the zero counting formula

$$N = \frac{1}{2\pi i}\int_{C_r} \frac{f'(z)}{f(z)}\,dz$$

to give yet another proof of the Fundamental Theorem of Algebra. (This proof will show that a polynomial of degree N has N roots, counted with multiplicity.)

2. Prove that an isolated singularity of $f(z)$ is removable as soon as either Re $f(z)$ or Im $f(z)$ is bounded above or below near the singularity.

3. Suppose that f_n is a sequence of analytic functions on a domain Ω containing $\{z : |z| \le 1\}$ and suppose that f_n is uniformly Cauchy on the set $\{z : |z| = 1\}$. Show that f_n converges uniformly on $\{z : |z| < 1\}$ to a function f which is analytic there.

4. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.

5. Prove that if h is an analytic branch of $f^{1/n}$, then $h'/h = f'/(nf)$.

6. Derive the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n}\theta\; d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

by integrating

$$\frac{1}{z}\left(z+\frac{1}{z}\right)^{2n}$$

around the unit circle.