Math 530

Homework 7

1. Find a one-to-one conformal mapping of the region common to the two disks $|z-1| < \sqrt{2}$ and $|z+1| < \sqrt{2}$ onto the unit disk.

2. Find a one-to-one conformal mapping of the region $\{z : 0 < \text{Re } z < 1\}$ onto the unit disk.

3. Let Ω denote the open set obtained by removing from $\mathbb{C}$ the interval $[-1, 1]$. Prove that there is an analytic function $F(z)$ on Ω such that $F(z)^2 = \frac{z+1}{z-1}$. *Hint:* What is the image of Ω under the map $(z+1)/(z-1)$?

4. Show that the Laurent expansion of $(e^z - 1)^{-1}$ at the origin is of the form $\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$. The numbers B_k are known as the Bernoulli numbers. Calculate B_1 and B_2.

5. Prove that a Laurent series can be differentiated term by term. When can a Laurent series be anti-differentiated term by term?

6. Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - 1| < 1$ in a domain Ω. Prove that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for every closed curve in Ω.

7. Suppose that f_n is a sequence of analytic functions on a domain Ω which converges uniformly on compact subsets of Ω to a non-constant function f. Suppose that f has a zero of order m at a point a in Ω. Prove that there is an $\epsilon > 0$ and a positive integer N such that each function $f_n(z)$ with $n > N$ has exactly m zeroes (counted with multiplicity) on $D_\epsilon(a) \subset \Omega$.

8. Suppose that f_n is a sequence of analytic functions on a domain Ω which converges uniformly on compact subsets of Ω to a function f. Suppose that $\widetilde{\Omega}$ is a domain containing $f_n(\Omega)$ for each n. Prove that, if f is not constant, then $\widetilde{\Omega}$ contains $f(\Omega)$ too.

9. Compute

a) $\displaystyle\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx,$ b) $\displaystyle\int_0^\infty \frac{1}{1+x^5}\, dx,$

c) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\, dx,$ a real, d) $\displaystyle\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^2 dx.$